Quantizations of generalized-Witt algebra and of Jacobson-Witt algebra in the modular case

Naihong Hu* and Xiuling Wang

To the memory of Professor Xihua Cao

Abstract. We quantize the generalized-Witt algebra in characteristic 0 with its Lie bialgebra structures discovered by Song-Su ([**10**]). Via a modulo p reduction and a modulo "p-restrictedness" reduction process, we get 2^n-1 families of truncated p-polynomial noncocommutative deformations of the restricted universal enveloping algebra of the Jacobson-Witt algebra $\mathbf{W}(n;\underline{1})$ (for the Cartan type simple modular restricted Lie algebra of W type). They are new families of noncommutative and noncocommutative Hopf algebras of dimension p^{1+np^n} in characteristic p. Our results generalize a work of Grunspan (J. Algebra 280 (2004), 145–161, [**6**]) in rank $n=1$ case in characteristic 0. In the modular case, the argument for a refined version follows from the modular reduction approach (different from [**6**]) with some techniques from the modular Lie algebra theory.

1. Introduction and Definitions

In a paper by Michaelis ([**7**]) a class of infinite-dimensional Lie bialgebras containing the Virasoro algebra was presented. Afterwards, this type of Lie bialgebras was further classified by Ng and Taft ([**8**]). Recently, Song and Su ([**10**]) classified all Lie bialgebra structures on a given Lie algebra of generalized Witt type ([**2**]), which turned out to be coboundary triangular (for the definition, see p. 28, [**4**]).

In Hopf algebra or quantum group theory, two standard methods to yield new bialgebras from old ones are by twisting the product by a 2-cocycle but keeping the coproduct unchanged, and by twisting the coproduct by a Drinfel'd twist but preserving the product. Constructing quantizations of Lie bialgebras is an important approach to producing new quantum groups (see [**3**], [**4**] and references therein). Recently, Grunspan [**6**] obtained the quantizations of the (infinite-dimensional) Witt algebra W in characteristic 0 by using the twist discovered by Giaquinto and Zhang [**5**], and of its simple modular Witt algebra $\mathbf{W}(1;\underline{1})$ of dimension p in characteristic p by a reduction modulo p. In the modular case, however, his treatment did not

1991 *Mathematics Subject Classification.* Primary 17B37, 17B62; Secondary 17B50.

Key words and phrases. Quantization, (basic) Drinfel'd twist, Lie bialgebra, generalized-Witt algebra, Jacobson-Witt algebra, Hopf algebra.

*N.H., Corresponding author, supported by the NNSF (Grant 10431040), the PCSIRT, the TRAPOYT and the FUDP from the MOE of China, and the SRSTP from the STCSM.

work for the restricted universal enveloping algebra $\mathbf{u}(\mathbf{W}(1;\underline{1}))$. One reason is that the periodic quotient with respect to the ideal J he adopted (see Definition 5, p. 158 [**6**]) resulted in some extra relations like $[e_{p-2}, e_2] = 4e_p \equiv 4e_o \neq 0$ incorrectly imposed on it. So in this case we need to consider the best way to approach it. Another reason is that the assertion in Theorem 2 [**6**] (a corrected version) is valid only for one specific case $i = 1$ (except for the other cases $i = 2, \cdots, p-1$). This means that just one family (rather than $p-1$ families asserted in [**6**]) of polynomial noncocommutative deformations of $\mathbf{u}(\mathbf{W}(1;\underline{1}))$ is able to be obtained via a modular reduction process. The correct modular reduction approach should be clarified as stated below.

In this paper, we first use the general quantization method by a Drinfel'd twist (cf. [**1**]) to quantize explicitly the newly defined triangular Lie bialgebra structure on the generalized-Witt algebra in characteristic 0 ([**10**]). Actually, this process completely depends on the construction of Drinfel'd twists which, up to integral scalars, are controlled by the classical Yang-Baxter r-matrix. To study the modular case in characteristic p, we work over the so-called *"positive"* part subalgebra $\mathbf{W}^+$ of the generalized-Witt algebra $\mathbf{W}$. It is an infinite-dimensional simple Lie algebra when defined over a field of characteristic 0, while, defined over a field of characteristic p, it contains a maximal ideal $J_{\underline{1}}$ and the corresponding quotient is exactly the Jacobson-Witt algebra $\mathbf{W}(n;\underline{1})$, which is a Cartan type restricted simple modular Lie algebra of W type. In order to yield the *expected* finite-dimensional quantizations of the restricted universal enveloping algebra for the Jacobson-Witt algebra $\mathbf{W}(n;\underline{1})$, in concept at first, we need to give a precise definition concerning what is the quantization of the above object in the modular case (see Definition 3.3), and then go on two steps of our reduction: *modulo p reduction* and *modulo "p-restrictedness" reduction*. As a crucial immediate step for the *modulo p reduction*, we need first to develop the quantization integral forms for the $\mathbb{Z}$-form $\mathbf{W}^+_{\mathbb{Z}}$ in characteristic 0. In this case, a new phenomena appeared is that there exist n the so-called *basic Drinfel'd twists* whose pairwise different products among them afford the possible Drinfel'd twists (see Proposition 2.12). Accordingly, we get 2^n-1 new quantization integral forms for $\mathbf{W}^+_{\mathbb{Z}}$ in characteristic 0, which, via the *modulo "p-restrictedness" reduction*, eventually lead to 2^n-1 new examples of Hopf algebra of dimension p^{1+np^n} with indeterminate t, or of dimension p^{np^n} with specializing t into a scalar in $\mathcal{K}$ in characteristic p.

A remarkable point is that these Hopf algebras we obtained contain the well-known Radford algebra as a Hopf subalgebra. Our work extends the class of examples of noncommutative and noncocommutative finite-dimensional Hopf algebras in finite characteristic (see [**13**]).

1.1. Generalized-Witt algebra and its Lie bialgebra structure. Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) = 0$ and let $n > 0$. The notations used here are the same as in [**2**]. Let $\mathbb{F}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$ be a Laurent polynomial algebra, and ∂_i coincides with the degree operator $x_i\frac{\partial}{\partial x_i}$. Set $T = \bigoplus_{i=1}^n \mathbb{Z}\partial_i$. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$, write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Denote $\mathbf{W} = \mathbb{F}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \otimes_{\mathbb{Z}} T = \mathrm{Span}_{\mathbb{F}}\{\, x^\alpha\partial \mid \alpha \in \mathbb{Z}^n, \partial \in T\}$, where we set $x^\alpha\partial\ = x^\alpha \otimes \partial$ for short. Then $\mathbf{W} = \mathrm{Der}_{\mathbb{F}}(\mathbb{F}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}])$ is a Lie algebra of generalized-Witt type (for definition, see [**2**]) under the following bracket

$$[\, x^\alpha\partial,\, x^\beta\partial'\,] = x^{\alpha+\beta}\big(\,\partial(\beta)\partial' - \partial'(\alpha)\partial\,\big), \qquad \forall\ \alpha,\, \beta \in \mathbb{Z}^n;\ \partial,\, \partial' \in T,$$

where $\partial(\beta) = \langle \partial, \beta\rangle = \langle \beta, \partial\rangle = \sum\limits_{i=1}^{n} a_i\beta_i \in \mathbb{Z}$ for $\partial = \sum\limits_{i=1}^{n} a_i\partial_i \in T$ and $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{Z}^n$. The bilinear map $\langle \cdot, \cdot \rangle : T \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$ is non-degenerate in the sense: $\partial(\alpha) = \langle \partial, \alpha\rangle = 0$ $(\forall\ \partial \in T)$, $\Longrightarrow \alpha = 0$, and $\partial(\alpha) = \langle \partial, \alpha\rangle = 0$ $(\forall\ \alpha \in \mathbb{Z}^n)$, $\Longrightarrow \partial = 0$. $\mathbf{W}$ is an infinite dimensional simple Lie algebra over $\mathbb{F}$.

The following result is due to [**10**].

PROPOSITION 1.1. *There is a triangular Lie bialgebra structure on* $\mathbf{W}$ *given by the classical Yang-Baxter r-matrix* $r := \partial_0 \otimes x^\gamma \partial_0' - x^\gamma\partial_0' \otimes \partial_0$, *for* $\partial_0,\ \partial_0' \in T$, $\gamma \in \mathbb{Z}^n$, *where* $[\,\partial_0, x^\gamma\partial_0'\,] = \partial_0(\gamma)x^\gamma\partial_0'$.

From Proposition 1.1, we notice that the classical Yang-Baxter r-matrix r is uniquely expressed as the antisymmetric tensor of *two distinguished elements* ∂_0, $x^\gamma\partial_0'$ up to scalars satisfying $[\,\partial_0, x^\gamma\partial_0'\,] = \partial_0(\gamma)x^\gamma\partial_0'$. In fact, for a given r-matrix, we may take *two distinguished elements* of the form $h := \partial_0(\gamma)^{-1}\partial_0$ and $e := \partial_0(\gamma)x^\gamma\partial_0'$ such that $[h, e] = e$, where $\partial_0(\gamma) \neq 0$ $(\in \mathbb{Z})$.

1.2. Generalized-Witt subalgebra $\mathbf{W}^+$. Denote $D_i = \frac{\partial}{\partial x_i}$. Set $\mathbf{W}^+ := \text{Span}_{\mathcal{K}}\{x^\alpha D_i \mid \alpha \in \mathbb{Z}_+^n, 1 \leq i \leq n\}$, where $\mathbb{Z}_+$ is the set of non-negative integers. Then $\mathbf{W}^+ = \text{Der}_{\mathcal{K}}(\mathcal{K}[x_1, \cdots, x_n])$ is the derivation Lie algebra of polynomial ring $\mathcal{K}[x_1, \cdots, x_n]$, which, via the identification $x^\alpha D_i$ with $x^{\alpha-\epsilon_i}\partial_i$, can be considered as a Lie subalgebra (the "*positive*" part) of the generalized-Witt algebra $\mathbf{W}$ over a field $\mathcal{K}$. Evidently, we have the following result (see Ex. 8, p. 153 in [**12**])

LEMMA 1.2. (1) *If* $\mathcal{K} = \mathbb{F}$, *i.e., char*$(\mathcal{K}) = 0$, *then* $\mathbf{W}^+$ *is a simple Lie algebra of infinite dimension, which is a Lie subalgebra of* $\mathbf{W}$.

(2) *If char*$(\mathcal{K}) = p$, *then there exists a maximal ideal* $J_{\underline{1}} := \langle\{x^\alpha D_i \mid \exists j : \alpha_j \geq p, 1 \leq i \leq n\}\rangle$ *in* $\mathbf{W}^+$ *such that* $\mathbf{W}^+/J_{\underline{1}} \cong \mathbf{W}(n;\underline{1})$ *under the identification* $\frac{1}{\alpha!}x^\alpha D_i$ $(0 \leq \alpha \leq \tau)$ *with* $x^{(\alpha)}D_i$ *and the others with* 0, *where* $\mathbf{W}(n;\underline{1})$ *is the Jacobson-Witt algebra defined in the subsection below.*

1.3. Jacobson-Witt algebra $\mathbf{W}(n;\underline{1})$. Assume now that $\text{char}(\mathcal{K}) = p$, then by definition (cf. [**11**]), the Jacobson-Witt algebra $\mathbf{W}(n;\underline{1})$ is a restricted simple Lie algebra over a field $\mathcal{K}$. Its structure of p-Lie algebra is given by $D^{[p]} = D^p$, $\forall\, D \in \mathbf{W}(n;\underline{1})$ with a basis $\{\, x^{(\alpha)}D_j \mid 1 \leq j \leq n,\ 0 \leq \alpha \leq \tau\}$, where $\underline{1} = (1, \cdots, 1)$, $\tau = (p-1, \cdots, p-1) \in \mathbb{N}^n$; $\epsilon_i = (\delta_{1i}, \cdots, \delta_{ni})$ such that $x^{(\epsilon_i)} = x_i$ and $D_j(x_i) = \delta_{ij}$; and $\mathcal{O}(n;\underline{1}) := \{\, x^{(\alpha)} \mid 0 \leq \alpha \leq \tau\}$ is the restricted divided power algebra with $x^{(\alpha)}x^{(\beta)} = \binom{\alpha+\beta}{\alpha}x^{(\alpha+\beta)}$ and a convention: $x^{(\alpha)} = 0$ if α has a component $\alpha_j < 0$ or $> p$, where $\binom{\alpha+\beta}{\alpha} := \prod_{i=1}^n \binom{\alpha_i+\beta_i}{\alpha_i}$. Moreover, $\mathbf{W}(n;\underline{1}) \cong \text{Der}_{\mathcal{K}}(\mathcal{O}(n;\underline{1}))$.

By definition, the restricted universal enveloping algebra $\mathbf{u}(\mathbf{W}(n;\underline{1}))$ is isomorphic to $U(\mathbf{W}(n;\underline{1}))/I$ where I is the Hopf ideal of $U(\mathbf{W}(n;\underline{1}))$ generated by $H_i^p - H_i$, D^p with $D \neq H_i = x^{(\epsilon_i)}D_i$. Since $dim_{\mathcal{K}}\mathbf{W}(n;\underline{1}) = np^n$, we have $dim_{\mathcal{K}}\mathbf{u}(\mathbf{W}(n;\underline{1})) = p^{np^n}$.

1.4. Quantization by Drinfel'd twists. The following result is well-known (see [**1**], [**3**], [**4**], etc.).

LEMMA 1.3. *Let* $(A, m, \iota, \Delta_0, \varepsilon_0, S_0)$ *be a Hopf algebra over a commutative ring. A Drinfel'd twist* $\mathcal{F}$ *on* A *is an invertible element of* $A \otimes A$ *such that*

$$(\mathcal{F} \otimes 1)(\Delta_0 \otimes \text{Id})(\mathcal{F}) = (1 \otimes \mathcal{F})(\text{Id} \otimes \Delta_0)(\mathcal{F}),$$
$$(\varepsilon_0 \otimes \text{Id})(\mathcal{F}) = 1 = (\text{Id} \otimes \varepsilon_0)(\mathcal{F}).$$

Then, $w = m(\mathrm{Id}\otimes S_0)(\mathcal{F})$ *is invertible in* A *with* $w^{-1} = m(S_0\otimes \mathrm{Id})(\mathcal{F}^{-1})$.

Moreover, if we define $\Delta: A \longrightarrow A\otimes A$ *and* $S: A\longrightarrow A$ *by*

$$\Delta(a) = \mathcal{F}\Delta_0(a)\mathcal{F}^{-1}, \qquad S = w\, S_0(a)\, w^{-1},$$

then $(A, m, \iota, \Delta, \varepsilon, S)$ *is a new Hopf algebra, called the twisting of* A *by the Drinfel'd twist* $\mathcal{F}$.

Let $\mathbb{F}[[t]]$ be a ring of formal power series over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$. Assume that L is a triangular Lie bialgebra over $\mathbb{F}$ with a classical Yang-Baxter r-matrix r (see [**3**], [**4**]). Let $U(L)$ be the universal enveloping algebra of L, with the standard Hopf algebra structure $(U(L), m, \iota, \Delta_0, \varepsilon_0, S_0)$.

Now let us consider *the topologically free* $\mathbb{F}[[t]]$*-algebra* $U(L)[[t]]$ (for definition, see p. 4, [**4**]), which can be viewed as an associative $\mathbb{F}$-algebra of formal power series with coefficients in $U(L)$. Naturally, $U(L)[[t]]$ equips with an induced Hopf algebra structure arising from that on $U(L)$. By abuse of notation, we denote it by $(U(L)[[t]], m, \iota, \Delta_0, \varepsilon_0, S_0)$.

Definition 1.4. For a triangular Lie bialgebra L over $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$, $U(L)[[t]]$ is called *a quantization of* $U(L)$ *by a Drinfel'd twist* $\mathcal{F}$ over $U(L)[[t]]$ if $U(L)[[t]]/tU(L)[[t]] \cong U(L)$, and $\mathcal{F}$ is determined by its r-matrix r (namely, its Lie bialgebra structure).

1.5. A crucial Lemma. For any element x of a unital R-algebra (R a ring) and $a\in R$, we set (see [**5**])

$$x_a^{\langle n\rangle} := (x+a)(x+a+1)\cdots(x+a+n-1) \tag{1}$$

and $x^{\langle n\rangle} := x_0^{\langle n\rangle}$.

We also set

$$x_a^{[n]} := (x+a)(x+a-1)\cdots(x+a-n+1) \tag{2}$$

and $x^{[n]} := x_0^{[n]}$.

Lemma 1.5. ([**5**], [**6**]) *For any element* x *of a unital* $\mathbb{F}$*-algebra with char*$(\mathbb{F}) = 0$, $a, b\in\mathbb{F}$ *and* $r, s, t\in\mathbb{Z}$, *one has*

$$x_a^{\langle s+t\rangle} = x_a^{\langle s\rangle}\, x_{a+s}^{\langle t\rangle}, \tag{3}$$

$$x_a^{[s+t]} = x_a^{[s]}\, x_{a-s}^{[t]}, \tag{4}$$

$$x_a^{[s]} = x_{a-s+1}^{\langle s\rangle}, \tag{5}$$

$$\sum_{s+t=r} \frac{(-1)^t}{s!\,t!}\, x_a^{[s]}\, x_b^{\langle t\rangle} = \binom{a-b}{r} = \frac{(a-b)\cdots(a-b-r+1)}{r!}, \tag{6}$$

$$\sum_{s+t=r} \frac{(-1)^t}{s!\,t!}\, x_a^{[s]}\, x_{b-s}^{[t]} = \binom{a-b+r-1}{r} = \frac{(a-b)\cdots(a-b+r-1)}{r!}. \tag{7}$$

2. Quantization of Lie bialgebra of generalized-Witt type

2.1. Some commutative relations in $U(\mathbf{W})$. For the universal enveloping algebra $U(\mathbf{W})$ of the generalized-Witt algebra $\mathbf{W}$ over $\mathbb{F}$, we need to do some necessary calculations, which are important for the construction of the Drinfel'd twists in the sequel.

LEMMA 2.1. *Fix the two distinguished elements* $h := \partial_0(\gamma)^{-1}\partial_0$ *and* $e := \partial_0(\gamma)x^\gamma\partial_0'$ *with* $\partial_0(\gamma) \in \mathbb{Z}^*$ *for* $\mathbf{W}$. *For* $a \in \mathbb{F}$, $\alpha \in \mathbb{Z}^n$, *and* m, k *non-negative integers, the following equalities hold in* $U(\mathbf{W})$:

$$x^\alpha\partial \cdot h_a^{[m]} = h^{[m]}_{a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}} \cdot x^\alpha\partial, \tag{8}$$

$$x^\alpha\partial \cdot h_a^{\langle m\rangle} = h^{\langle m\rangle}_{a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}} \cdot x^\alpha\partial, \tag{9}$$

$$e^k \cdot h_a^{[m]} = h_{a-k}^{[m]} \cdot e^k, \tag{10}$$

$$e^k \cdot h_a^{\langle m\rangle} = h_{a-k}^{\langle m\rangle} \cdot e^k, \tag{11}$$

$$x^\alpha\partial \cdot (x^\beta\partial')^m = \sum_{\ell=0}^{m}(-1)^\ell\binom{m}{\ell}(x^\beta\partial')^{m-\ell} \cdot x^{\alpha+\ell\beta}\Big(a_\ell\partial - b_\ell\partial'\Big), \tag{12}$$

where $a_\ell = \prod_{j=0}^{\ell-1}\partial'(\alpha+j\beta)$, $b_\ell = \ell\,\partial(\beta)a_{\ell-1}$, *and set* $a_0 = 1$, $b_0 = 0$.

PROOF. One has $x^\alpha\partial \cdot \partial_0 = \partial_0 \cdot x^\alpha\partial - \partial_0(\alpha)\,x^\alpha\partial$. So it is easy to see that (8) is true for $m = 1$. Suppose that (8) is true for m, then

$$\begin{aligned}
x^\alpha\partial \cdot h_a^{[m+1]} &= x^\alpha\partial \cdot h_a^{[m]} \cdot h_{a-m} \\
&= h^{[m]}_{a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}} \cdot x^\alpha\partial \cdot (h + a - m) \\
&= h^{[m]}_{a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}} \cdot \Big(\partial_0(\gamma)^{-1}(\partial_0 \cdot x^\alpha\partial - \partial_0(\alpha)x^\alpha\partial) + (a-m)x^\alpha\partial\Big) \\
&= h^{[m+1]}_{a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}} \cdot x^\alpha\partial.
\end{aligned}$$

Hence (8) holds for all m. Similarly, we can get (9), (10) and (11) by induction.

Formula (12) is a consequence of the fact (see Proposition 1.3 (4), [**12**]) that for any elements a, c in an associative algebra, one has

$$c\,a^m = \sum_{\ell=0}^{m}(-1)^\ell\binom{m}{\ell}a^{m-\ell}(\operatorname{ad} a)^\ell(c),$$

together with the formula

$$(\operatorname{ad} x^\beta\partial')^\ell(x^\alpha\partial) = x^{\alpha+\ell\beta}(a_\ell\partial - b_\ell\partial'), \tag{13}$$

obtained by induction on ℓ when taking $a = x^\beta\partial'$, $c = x^\alpha\partial$. □

Let us denote by $(U(\mathbf{W}), m, \iota, \Delta_0, S_0, \varepsilon_0)$ the standard Hopf algebra structure on $U(\mathbf{W})$, *i.e.*, we have the definitions of the coproduct, the antipode and the counit as follows

$$\begin{aligned}
\Delta_0(x^\alpha\partial) &= x^\alpha\partial \otimes 1 + 1 \otimes x^\alpha\partial, \\
S_0(x^\alpha\partial) &= -x^\alpha\partial, \\
\varepsilon_0(x^\alpha\partial) &= 0.
\end{aligned}$$

2.2. Quantization of $U(\mathbf{W})$ in characteristic 0. To describe a quantization of $U(\mathbf{W})$ by a Drinfel'd twist $\mathcal{F}$ over $U(\mathbf{W})[[t]]$, we need to construct explicitly such a Drinfel'd twist. In what follows, we shall see that such a twist in our case depends heavily upon the choice of *two distinguished elements* h, e arising from its r-matrix r (see subsection 1.1).

Now we proceed with the construction. For $a \in \mathbb{F}$, we set

$$\mathcal{F}_a = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_a^{[r]} \otimes e^r t^r, \qquad F_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{\langle r\rangle} \otimes e^r t^r,$$
$$u_a = m \cdot (S_0 \otimes \mathrm{Id})(F_a), \qquad v_a = m \cdot (\mathrm{Id} \otimes S_0)(\mathcal{F}_a).$$

Write $\mathcal{F} = \mathcal{F}_0$, $F = F_0$, $u = u_0$, $v = v_0$.

Since $S_0(h_a^{\langle r\rangle}) = (-1)^r h_{-a}^{[r]}$ and $S_0(e^r) = (-1)^r e^r$, we obtain

$$v_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{[r]} e^r t^r, \quad u_b = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_{-b}^{[r]} e^r t^r.$$

Lemma 2.2. *For $a, b \in \mathbb{F}$, one has*

$$\mathcal{F}_a F_b = 1 \otimes (1 - et)^{a-b}, \quad \textit{and} \quad v_a u_b = (1 - et)^{-(a+b)}.$$

Proof. Using Lemma 1.5, we obtain

$$\begin{aligned}
\mathcal{F}_a F_b &= \sum_{r,s=0}^{\infty} \frac{(-1)^r}{r!s!} h_a^{[r]} h_b^{\langle s\rangle} \otimes e^r e^s t^r t^s \\
&= \sum_{m=0}^{\infty} (-1)^m \left(\sum_{r+s=m} \frac{(-1)^s}{r!s!} h_a^{[r]} h_b^{\langle s\rangle} \right) \otimes e^m t^m \\
&= \sum_{m=0}^{\infty} (-1)^m \binom{a-b}{m} \otimes e^m t^m = 1 \otimes (1 - et)^{a-b}. \\
v_a u_b &= \sum_{m,n=0}^{\infty} \frac{(-1)^m}{m!n!} h_a^{[n]} e^n h_{-b}^{[m]} e^m t^{m+n} \\
&= \sum_{r=0}^{\infty} \sum_{m+n=r} \frac{(-1)^m}{m!n!} h_a^{[n]} h_{-b-n}^{[m]} e^r t^r \\
&= \sum_{r=0}^{\infty} \binom{a+b+r-1}{r} e^r t^r = (1 - et)^{-(a+b)}.
\end{aligned}$$

This completes the proof. □

Corollary 2.3. For $a \in \mathbb{F}$, $\mathcal{F}_a$ and u_a are invertible with $\mathcal{F}_a^{-1} = F_a$ and $u_a^{-1} = v_{-a}$. In particular, $\mathcal{F}^{-1} = F$ and $u^{-1} = v$.

Lemma 2.4. *For any positive integers r, we have*

$$\Delta_0(h^{[r]}) = \sum_{i=0}^{r} \binom{r}{i} h^{[i]} \otimes h^{[r-i]}.$$

Furthermore, $\Delta_0(h^{[r]}) = \sum_{i=0}^{r} \binom{r}{i} h_{-s}^{[i]} \otimes h_s^{[r-i]}$ *for any* $s \in \mathbb{F}$.

Proof. By induction on r, it is easy to see that it is true for $r = 1$. If it is true for r, then

$$\begin{aligned}
\Delta_0(h^{[r+1]}) &= \left(\sum_{i=0}^{r}\binom{r}{i}h^{[i]}\otimes h^{[r-i]}\right)\Big((h-r)\otimes 1 + 1\otimes(h-r) + r(1\otimes 1)\Big)\\
&= \left(\sum_{i=1}^{r-1}\binom{r}{i}h^{[i]}\otimes h^{[r-i]}\right)((h-r)\otimes 1 + 1\otimes(h-r))\\
&\quad + r\left(\sum_{i=0}^{r}\binom{r}{i}h^{[i]}\otimes h^{[r-i]}\right) + 1\otimes h^{[r+1]} + (h-r)\otimes h^{[r]}\\
&\quad + h^{[r+1]}\otimes 1 + h^{[r]}\otimes(h-r)\\
&= 1\otimes h^{[r+1]} + h^{[r+1]}\otimes 1 + r\left(\sum_{i=1}^{r-1}\binom{r}{i}h^{[i]}\otimes h^{[r-i]}\right)\\
&\quad + h\otimes h^{[r]} + h^{[r]}\otimes h + \sum_{i=1}^{r-1}\binom{r}{i}h^{[i+1]}\otimes h^{[r-i]}\\
&\quad + \sum_{i=1}^{r-1}(i-r)\binom{r}{i}h^{[i]}\otimes h^{[r-i]} + \sum_{i=1}^{r-1}\binom{r}{i}h^{[i]}\otimes h^{[r-i+1]}\\
&\quad + \sum_{i=1}^{r-1}(-i)\binom{r}{i}h^{[i]}\otimes h^{[r-i]}\\
&= 1\otimes h^{[r+1]} + h^{[r+1]}\otimes 1 + \sum_{i=1}^{r}\left[\binom{r}{i-1}+\binom{r}{i}\right]h^{[i]}\otimes h^{[r-i+1]}\\
&= \sum_{i=0}^{r+1}\binom{r+1}{i}h^{[i]}\otimes h^{[r+1-i]}.
\end{aligned}$$

Therefore, the formula holds by induction. $\square$

Proposition 2.5. *$\mathcal{F} = \sum\limits_{r=0}^{\infty}\frac{(-1)^r}{r!}h^{[r]}\otimes e^r t^r$ is a Drinfel'd twist on $U(\mathbf{W})[[t]]$, that is to say, the equalities hold*

$$(\mathcal{F}\otimes 1)(\Delta_0\otimes \mathrm{Id})(\mathcal{F}) = (1\otimes\mathcal{F})(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}),$$
$$(\varepsilon_0\otimes\mathrm{Id})(\mathcal{F}) = 1 = (\mathrm{Id}\otimes\varepsilon_0)(\mathcal{F}).$$

Proof. The second equality holds evidently. For the first one, by Lemmas 2.4 & 1.5, it is easy to get

$$\begin{aligned}
\mathrm{LHS} &= (\mathcal{F}\otimes 1)(\Delta_0\otimes\mathrm{Id})(\mathcal{F})\\
&= \left(\sum_{r=0}^{\infty}\frac{(-1)^r t^r}{r!}h^{[r]}\otimes e^r\otimes 1\right)\left(\sum_{s=0}^{\infty}\frac{(-1)^s t^s}{s!}\sum_{i=0}^{s}\binom{s}{i}h_{-r}^{[i]}\otimes h_r^{[s-i]}\otimes e^s\right)\\
&= \sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}t^{r+s}}{r!s!}\sum_{i=0}^{s}\binom{s}{i}h^{[r]}h_{-r}^{[i]}\otimes e^r h_r^{[s-i]}\otimes e^s\\
&= \sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}t^{r+s}}{r!s!}\sum_{i=0}^{s}\binom{s}{i}h^{[r+i]}\otimes h^{[s-i]}e^r\otimes e^s.
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= (1\otimes\mathcal{F})(\mathrm{Id}\otimes\Delta_0)(\mathcal{F})\\
&= \Big(\sum_{r=0}^{\infty}\frac{(-1)^r t^r}{r!}1\otimes h^{[r]}\otimes e^r\Big)\Big(\sum_{s=0}^{\infty}\frac{(-1)^s t^s}{s!}\sum_{i=0}^{s}\binom{s}{i}h^{[s]}\otimes e^i\otimes e^{s-i}\Big)\\
&= \sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}t^{r+s}}{r!s!}\sum_{i=0}^{s}\binom{s}{i}h^{[s]}\otimes h^{[r]}e^i\otimes e^{r+s-i}.
\end{aligned}$$

It suffices to show that

$$\sum_{p+q=m}\frac{1}{p!q!}\sum_{j=0}^{p}\binom{p}{j}h^{[q+j]}\otimes h^{[p-j]}e^q\otimes e^p=\sum_{r+s=m}\frac{1}{r!s!}\sum_{i=0}^{s}\binom{s}{i}h^{[s]}\otimes h^{[r]}e^i\otimes e^{m-i}.$$

Fix r, s, i such that $r+s=m$, $0\le i\le s$. Set $q=i, q+j=s$, then $p=m-i$, $p-j=r$. We can see that the coefficients of $h^{[s]}\otimes h^{[r]}e^i\otimes e^{m-i}$ in both sides are equal. $\square$

Having the above Proposition in hand, by Lemma 1.3, now we can perform the process of twisting the standard Hopf structure $(U(\mathbf{W})[[t]],\, m,\, \iota,\, \Delta_0,\, S_0,\, \varepsilon_0)$ by the Drinfel'd twist $\mathcal{F}$ constructed above.

The following Lemma is very useful to our main result in this section.

LEMMA 2.6. *For $a\in\mathbb{F}$, $\alpha\in\mathbb{Z}^n$, one has*

$$((x^\alpha\partial)^s\otimes 1)\cdot F_a=F_{a-s\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot((x^\alpha\partial)^s\otimes 1),\tag{14}$$

$$(1\otimes x^\alpha\partial)\cdot F_a=\sum_{\ell=0}^{\infty}(-1)^\ell F_{a+\ell}\cdot\Big(h_a^{\langle\ell\rangle}\otimes x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')t^\ell\Big),\tag{15}$$

$$(x^\alpha\partial)\cdot u_a=u_{a+\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\Big(\sum_{\ell=0}^{\infty}x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')\cdot h_{1-a}^{\langle\ell\rangle}t^\ell\Big),\tag{16}$$

$$(x^\alpha\partial)^s\cdot u_a=u_{a+s\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\Big(\sum_{\ell=0}^{\infty}d^{(\ell)}\big((x^\alpha\partial)^s\big)\cdot h_{1-a}^{\langle\ell\rangle}t^\ell\Big),\tag{17}$$

$$(1\otimes(x^\alpha\partial)^s)\cdot F_a=\sum_{\ell=0}^{\infty}(-1)^\ell F_{a+\ell}\cdot\Big(h_a^{\langle\ell\rangle}\otimes d^{(\ell)}((x^\alpha\partial)^s)t^\ell\Big),\tag{18}$$

where $d^{(\ell)}:=\frac{1}{\ell!}(\mathrm{ad}\,e)^\ell$, $A_\ell=\frac{\partial_0(\gamma)^\ell}{\ell!}\prod_{j=0}^{\ell-1}\partial_0'(\alpha+j\gamma)$, $B_\ell=\partial_0(\gamma)\partial(\gamma)A_{\ell-1}$, and set $A_0=1$, $B_0=0$.

PROOF. For (14): By (9), one has

$$\begin{aligned}
(x^\alpha\partial\otimes 1)\cdot F_a &= \sum_{m=0}^{\infty}\frac{1}{m!}x^\alpha\partial\cdot h_a^{\langle m\rangle}\otimes e^m t^m\\
&= \sum_{m=0}^{\infty}\frac{1}{m!}h^{\langle m\rangle}_{a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot x^\alpha\partial\otimes e^m t^m\\
&= F_{a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot(x^\alpha\partial\otimes 1).
\end{aligned}$$

By induction on s, we obtain the result.

For (15): Let $a_\ell = \prod_{j=0}^{\ell-1} \partial_0'(\alpha+j\gamma)$, $b_\ell = \ell\,\partial(\gamma)a_{\ell-1}$. Then using (12) we get

$$\begin{aligned}
(1\otimes x^\alpha\partial)\cdot F_a &= \sum_{m=0}^{\infty}\frac{1}{m!}h_a^{\langle m\rangle}\otimes x^\alpha\partial\cdot e^m t^m \\
&= \sum_{m=0}^{\infty}\frac{1}{m!}h_a^{\langle m\rangle}\otimes\left(\sum_{\ell=0}^{m}(-1)^\ell\binom{m}{\ell}\partial_0(\gamma)^\ell e^{m-\ell}\cdot x^{\alpha+\ell\gamma}(a_\ell\partial-b_\ell\partial_0')t^m\right) \\
&= \sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}(-1)^\ell\frac{1}{m!\ell!}h_a^{\langle m+\ell\rangle}\otimes\partial_0(\gamma)^\ell e^m\cdot x^{\alpha+\ell\gamma}(a_\ell\partial-b_\ell\partial_0')t^{m+\ell} \\
&= \sum_{\ell=0}^{\infty}(-1)^\ell\left(\sum_{m=0}^{\infty}\frac{1}{m!}h_{a+\ell}^{\langle m\rangle}\otimes e^m t^m\right)\left(h_a^{\langle\ell\rangle}\otimes x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')t^\ell\right) \\
&= \sum_{\ell=0}^{\infty}(-1)^\ell F_{a+\ell}\cdot\left(h_a^{\langle\ell\rangle}\otimes x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')t^\ell\right).
\end{aligned}$$

For (16): Using (8), (10) and (12), we get

$$\begin{aligned}
x^\alpha\partial\cdot u_a &= x^\alpha\partial\cdot\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h_{-a}^{[r]}\cdot e^r t^r\right) \\
&= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}x^\alpha\partial\cdot h_{-a}^{[r]}\cdot e^r t^r \\
&= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h^{[r]}_{-a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot x^\alpha\partial\cdot e^r t^r \\
&= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h^{[r]}_{-a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\left(\sum_{\ell=0}^{r}(-1)^\ell\binom{r}{\ell}\partial_0(\gamma)^\ell e^{r-\ell}\cdot x^{\alpha+\ell\gamma}(a_\ell\partial-b_\ell\partial_0')t^r\right) \\
&= \sum_{r,\ell=0}^{\infty}\frac{(-1)^{r+\ell}}{(r+\ell)!}h^{[r+\ell]}_{-a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\left((-1)^\ell\binom{r+\ell}{\ell}\partial_0(\gamma)^\ell e^r\cdot x^{\alpha+\ell\gamma}(a_\ell\partial-b_\ell\partial_0')t^{r+\ell}\right) \\
&= \sum_{r,\ell=0}^{\infty}\frac{(-1)^r}{r!\ell!}\partial_0(\gamma)^\ell h^{[r]}_{-a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot h^{[\ell]}_{-a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}-r}\cdot e^r\cdot x^{\alpha+\ell\gamma}(a_\ell\partial-b_\ell\partial_0')t^{r+\ell} \\
&= \sum_{\ell=0}^{\infty}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h^{[r]}_{-a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot e^r t^r\right)\cdot h^{[\ell]}_{-a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')t^\ell \\
&= u_{a+\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\sum_{\ell=0}^{\infty}h^{[\ell]}_{-a-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')t^\ell \\
&= u_{a+\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\sum_{\ell=0}^{\infty}x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')\cdot h^{[\ell]}_{-a+\ell}t^\ell \\
&= u_{a+\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\sum_{\ell=0}^{\infty}x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')\cdot h^{\langle\ell\rangle}_{-a+1}t^\ell.
\end{aligned}$$

As for (17): By (13), we get

$$d^{(\ell)}(x^\alpha\partial) = x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0'). \tag{19}$$

Combining with (16), we get that (17) is true for $s=1$. Using the derivation property of $d^{(\ell)}$, we easily obtain that

$$d^{(\ell)}(a_1\cdots a_s)=\sum_{\ell_1+\cdots+\ell_s=\ell} d^{(\ell_1)}(a_1)\cdots d^{(\ell_s)}(a_s). \tag{20}$$

By induction on s, we have

$$\begin{aligned}
(x^\alpha\partial)^{s+1}\cdot u_a &= x^\alpha\partial\cdot u_{a+s\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\sum_{n=0}^{\infty} d^{(n)}((x^\alpha\partial)^s)\cdot h_{1-a}^{\langle n\rangle}t^n\\
&= u_{a+(s+1)\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\Big(\sum_{m=0}^{\infty} d^{(m)}(x^\alpha\partial)\cdot h^{\langle m\rangle}_{1-a-s\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}t^m\Big)\\
&\qquad\cdot\Big(\sum_{n=0}^{\infty} d^{(n)}((x^\alpha\partial)^s)\cdot h_{1-a}^{\langle n\rangle}t^n\Big)\\
&= u_{a+(s+1)\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\Big(\sum_{m,n=0}^{\infty} d^{(m)}(x^\alpha\partial)d^{(n)}((x^\alpha\partial)^s)h_{1-a+n}^{\langle m\rangle}h_{1-a}^{\langle n\rangle}t^{n+m}\Big)\\
&= u_{a+(s+1)\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\Big(\sum_{\ell=0}^{\infty}\sum_{m+n=\ell} d^{(m)}(x^\alpha\partial)d^{(n)}((x^\alpha\partial)^s)h_{1-a}^{\langle \ell\rangle}t^{\ell}\Big)\\
&= u_{a+(s+1)\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\Big(\sum_{\ell=0}^{\infty} d^{(\ell)}((x^\alpha\partial)^{s+1})h_{1-a}^{\langle \ell\rangle}t^{\ell}\Big),
\end{aligned}$$

where we get the first and second "=" by using the inductive hypothesis and (16), the third by using (13) & (20) and the fourth by using (4) & (20).

For (18): this follows from (15) & (20).

Thus, the proof is complete. □

The following theorem gives the quantization of $U(\mathbf{W})$ by the Drinfel'd twist $\mathcal{F}$, which is essentially determined by the Lie bialgebra triangular structure on $\mathbf{W}$.

THEOREM 2.7. *With the choice of two distinguished elements $h:=\partial_0(\gamma)^{-1}\partial_0$, $e:=\partial_0(\gamma)x^\gamma\partial_0'$ ($\partial_0(\gamma)\in\mathbb{Z}^*$) with $[h,e]=e$ in the generalized-Witt algebra $\mathbf{W}$ over $\mathbb{F}$, there exists a structure of noncommutative and noncocommutative Hopf algebra $(U(\mathbf{W})[[t]],m,\iota,\Delta,S,\varepsilon)$ on $U(\mathbf{W})[[t]]$ over $\mathbb{F}[[t]]$ with $U(\mathbf{W})[[t]]/tU(\mathbf{W})[[t]]\cong U(\mathbf{W})$, which leaves the product of $U(\mathbf{W})[[t]]$ undeformed but with the deformed coproduct, antipode and counit defined by*

$$\Delta(x^\alpha\partial)=x^\alpha\partial\otimes(1-et)^{\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}+\sum_{\ell=0}^{\infty}(-1)^\ell h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}\cdot x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')t^\ell, \tag{21}$$

$$S(x^\alpha\partial)=-(1-et)^{-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\Big(\sum_{\ell=0}^{\infty}x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')\cdot h_1^{\langle\ell\rangle}t^\ell\Big), \tag{22}$$

$$\varepsilon(x^\alpha\partial)=0, \tag{23}$$

where $\alpha\in\mathbb{Z}^n$, $A_\ell=\frac{\partial_0(\gamma)^\ell}{\ell!}\prod_{j=0}^{\ell-1}\partial_0'(\alpha+j\gamma)$, $B_\ell=\partial_0(\gamma)\partial(\gamma)A_{\ell-1}$, and set $A_0=1$, $B_0=0$.

Proof. By Lemmas 1.3 and 2.2, it follows from (14) and (15) that

$$\begin{aligned}\Delta(x^\alpha\partial) &= \mathcal{F}\cdot\Delta_0(x^\alpha\partial)\cdot\mathcal{F}^{-1}\\ &= \mathcal{F}\cdot(x^\alpha\partial\otimes 1)\cdot F+\mathcal{F}\cdot(1\otimes x^\alpha\partial)\cdot F\\ &= \Big(\mathcal{F}F_{-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\Big)\cdot(x^\alpha\partial\otimes 1)+\sum_{\ell=0}^{\infty}(-1)^\ell\Big(\mathcal{F}F_\ell\Big)\cdot\Big(h^{\langle\ell\rangle}\otimes x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')t^\ell\Big)\\ &= \Big(1\otimes(1-et)^{\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\Big)\cdot(x^\alpha\partial\otimes 1)\\ &\quad+\sum_{\ell=0}^{\infty}(-1)^\ell\Big(1\otimes(1-et)^{-\ell}\Big)\cdot\Big(h^{\langle\ell\rangle}\otimes x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')t^\ell\Big)\\ &= x^\alpha\partial\otimes(1-et)^{\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}+\sum_{\ell=0}^{\infty}(-1)^\ell h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}\cdot x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')t^\ell.\end{aligned}$$

By (16) and Lemma 2.2, we obtain

$$\begin{aligned}S(x^\alpha\partial) &= u^{-1}S_0(x^\alpha\partial)\,u=-v\cdot x^\alpha\partial\cdot u\\ &= -v\cdot u_{\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\Big(\sum_{\ell=0}^{\infty}x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')\cdot h_1^{\langle\ell\rangle}t^\ell\Big)\\ &= -(1-et)^{-\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\Big(\sum_{\ell=0}^{\infty}x^{\alpha+\ell\gamma}(A_\ell\partial-B_\ell\partial_0')\cdot h_1^{\langle\ell\rangle}t^\ell\Big).\end{aligned}$$

Hence, we get the result. □

For later use, we need to make the following

Lemma 2.8. *For $s\geq 1$, one has*

(i) $$\Delta((x^\alpha\partial)^s)=\sum_{\substack{0\leq j\leq s\\ \ell\geq 0}}\binom{s}{j}(-1)^\ell(x^\alpha\partial)^j h^{\langle\ell\rangle}\otimes(1-et)^{j\frac{\partial_0(\alpha)}{\partial_0(\gamma)}-\ell}d^{(\ell)}((x^\alpha\partial)^{s-j})t^\ell.$$

(ii) $$S((x^\alpha\partial)^s)=(-1)^s(1-et)^{-s\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\Big(\sum_{\ell=0}^{\infty}d^{(\ell)}((x^\alpha\partial)^s)\cdot h_1^{\langle\ell\rangle}t^\ell\Big).$$

Proof. By (18) and Lemma 2.2, we obtain

$$\begin{aligned}\Delta((x^\alpha\partial)^s) &= \mathcal{F}\Big(x^\alpha\partial\otimes 1+1\otimes x^\alpha\partial\Big)^s\mathcal{F}^{-1}\\ &= \sum_{j=0}^{s}\binom{s}{j}\mathcal{F}F_{-j\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}(x^\alpha\partial\otimes 1)^j\Big(\sum_{\ell\geq 0}(-1)^\ell\mathcal{F}F_\ell\Big(h^{\langle\ell\rangle}\otimes d^{(\ell)}((x^\alpha\partial)^{s-j})t^\ell\Big)\Big)\\ &= \sum_{j=0}^{s}\sum_{\ell\geq 0}\binom{s}{j}(-1)^\ell\Big((x^\alpha\partial)^j\otimes(1-et)^{j\frac{\partial_0(\alpha)}{\partial_0(\gamma)}-\ell}\Big)\Big(h^{\langle\ell\rangle}\otimes d^{(\ell)}((x^\alpha\partial)^{s-j})t^\ell\Big)\\ &= \sum_{\substack{0\leq j\leq s\\ \ell\geq 0}}\binom{s}{j}(-1)^\ell(x^\alpha\partial)^j h^{\langle\ell\rangle}\otimes(1-et)^{j\frac{\partial_0(\alpha)}{\partial_0(\gamma)}-\ell}d^{(\ell)}((x^\alpha\partial)^{s-j})t^\ell.\end{aligned}$$

Again by (17) and Lemma 2.2, we get

$$\begin{aligned} S((x^\alpha\partial)^s) &= u^{-1}S_0((x^\alpha\partial)^s)\,u = (-1)^s v\cdot(x^\alpha\partial)^s\cdot u \\ &= (-1)^s v\cdot u_{s\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\Big(\sum_{\ell=0}^{\infty} d^{(\ell)}((x^\alpha\partial)^s)\cdot h_1^{\langle\ell\rangle}t^\ell\Big) \\ &= (-1)^s(1-et)^{-s\frac{\partial_0(\alpha)}{\partial_0(\gamma)}}\cdot\Big(\sum_{\ell=0}^{\infty} d^{(\ell)}((x^\alpha\partial)^s)\cdot h_1^{\langle\ell\rangle}t^\ell\Big). \end{aligned}$$

This completes the proof. □

2.3. Quantization integral forms of $\mathbb{Z}$-form $\mathbf{W}_{\mathbb{Z}}^+$ in characteristic 0. Note that $\{x^\alpha D_i \mid \alpha\in\mathbb{Z}_+^n\}$ is a $\mathbb{Z}$-basis of $\mathbf{W}_{\mathbb{Z}}^+$. In order to get the quantization integral forms of $\mathbb{Z}$-form $\mathbf{W}_{\mathbb{Z}}^+$, it suffices to consider what conditions are for those coefficients occurred in the formulae (21) & (22) to be integral for the indicated basis elements.

Write $r=\partial_0(\gamma)$. Notice that suitable powers of factors $(1-et)^{\pm\frac{1}{r}}$ and coefficients A_ℓ and B_ℓ occur in (21) & (22). Grunspan ([**6**]) proved

Lemma 2.9. *For any $a,\,k,\,\ell\in\mathbb{Z}$, $a^\ell\prod_{j=0}^{\ell-1}(k+ja)/\ell!$ is an integer.*

According to this Lemma, we see that, if we take $\partial_0'=\partial_0$, then A_ℓ and B_ℓ are integers. However, those coefficients occurred in the expansions of $(1-et)^{\pm\frac{1}{r}}$ are not integral unless $r=1$. Consequently, the case we are interested in is only when $h=\partial_k$, $e=x^{\epsilon_k}\partial_k$ $(1\le k\le n)$, namely, $r=1$. Denote by $\mathcal{F}(k)$ the corresponding Drinfel'd twist. As a result of Theorem 2.7, we have

Corollary 2.10. With the specific choice of the two distinguished elements $h:=x^{\epsilon_k}D_k$, $e:=x^{2\epsilon_k}D_k$ $(1\le k\le n)$ with $[h,e]=e$, the corresponding quantization integral form of $U(\mathbf{W}_{\mathbb{Z}}^+)$ over $U(\mathbf{W}_{\mathbb{Z}}^+)[[t]]$ by the Drinfel'd twist $\mathcal{F}(k)$ with the algebra structure undeformed is given by

$$\Delta(x^\alpha D_i)=x^\alpha D_i\otimes(1-et)^{\alpha_k-\delta_{ik}}+\sum_{\ell=0}^{\infty}(-1)^\ell C_\ell\, h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}\cdot x^{\alpha+\ell\epsilon_k}D_it^\ell, \tag{24}$$

$$S(x^\alpha D_i)=-(1-et)^{-\alpha_k+\delta_{ik}}\cdot\Big(\sum_{\ell=0}^{\infty}C_\ell\, x^{\alpha+\ell\epsilon_k}D_i\cdot h_1^{\langle\ell\rangle}t^\ell\Big), \tag{25}$$

$$\varepsilon(x^\alpha D_i)=0, \tag{26}$$

where $\alpha\in\mathbb{Z}_+^n$, $C_\ell=A_\ell-B_\ell$, $A_\ell=\frac{1}{\ell!}\prod_{j=0}^{\ell-1}(\alpha_k-\delta_{ik}+j)$, $B_\ell=\delta_{ik}A_{\ell-1}$, and set $A_0=1$, $B_0=0$.

Remark 2.11. It is interesting to consider those products $\mathcal{F}(j_1)\cdots\mathcal{F}(j_s)$ of some pairwise different Drinfel'd twists $\mathcal{F}(j_1),\cdots,\mathcal{F}(j_s)$ with $j_1<\cdots<j_s$ in the system of the n so-called *basic Drinfel'd twists* $\{\,\mathcal{F}(1),\cdots,\mathcal{F}(n)\,\}$. Using the same argument as in the proof of Theorem 2.7, one can get many more new Drinfel'd twists (which depend on a bit more calculations to be carried out), which will not only lead to many more new quantization integral forms over the $U(\mathbf{W}_{\mathbb{Z}}^+)[[t]]$, but the possible quantizations over the $\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1}))$ as well, via our modular reduction approach developed in the next section.

More precisely, we note that $[\mathcal{F}(i),\mathcal{F}(j)]=0$ for any $1\le i,j\le n$. This fact, according to the definition of $\mathcal{F}(i)$, implies the commutative relations in the case $i\ne j$:

$$(*)\qquad \begin{aligned}(\mathcal{F}(j)\otimes 1)(\Delta_0\otimes \mathrm{Id})(\mathcal{F}(i))&=(\Delta_0\otimes \mathrm{Id})(\mathcal{F}(i))(\mathcal{F}(j)\otimes 1),\\ (1\otimes \mathcal{F}(j))(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(i))&=(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(i))(1\otimes\mathcal{F}(j)),\end{aligned}$$

which give rise to the following property.

Proposition 2.12. *$\mathcal{F}(i)\mathcal{F}(j)(i\ne j)$ is still a Drinfel'd twist on $U(\mathbf{W}_{\mathbb{Z}}^{+})[[t]]$. In general, $\mathcal{F}^\eta:=\mathcal{F}(1)^{\eta_1}\cdots\mathcal{F}(n)^{\eta_n}(\eta_i=0$ or $1)$ is a Drinfel'd twist on $U(\mathbf{W}_{\mathbb{Z}}^{+})[[t]]$.*

Proof. Note that $\Delta_0\otimes\mathrm{id}$, $\mathrm{id}\otimes\Delta_0$, $\varepsilon_0\otimes\mathrm{id}$ and $\mathrm{id}\otimes\varepsilon_0$ are algebraic homomorphisms. According to Proposition 2.5, it suffices to show that

$$(\mathcal{F}(i)\mathcal{F}(j)\otimes 1)(\Delta_0\otimes\mathrm{Id})(\mathcal{F}(i)\mathcal{F}(j))=(1\otimes\mathcal{F}(i)\mathcal{F}(j))(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(i)\mathcal{F}(j)).$$

Using $(*)$, we have

$$\begin{aligned}\mathrm{LHS}&=(\mathcal{F}(i)\otimes 1)(\mathcal{F}(j)\otimes 1)(\Delta_0\otimes\mathrm{Id})(\mathcal{F}(i))(\Delta_0\otimes\mathrm{Id})(\mathcal{F}(j))\\ &=(\mathcal{F}(i)\otimes 1)(\Delta_0\otimes\mathrm{Id})(\mathcal{F}(i))(\mathcal{F}(j)\otimes 1)(\Delta_0\otimes\mathrm{Id})(\mathcal{F}(j))\\ &=(1\otimes\mathcal{F}(i))(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(i))(1\otimes\mathcal{F}(j))(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(j))\\ &=(1\otimes\mathcal{F}(i))(1\otimes\mathcal{F}(j))(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(i))(\mathrm{Id}\otimes\Delta_0)(\mathcal{F}(j))=\mathrm{RHS}.\end{aligned}$$

As a result, $\mathcal{F}^\eta$ is also a Drinfel'd twist, which gives a quantization of $U(\mathbf{W}_{\mathbb{Z}}^{+})$. $\square$

Lemma 2.13. *With the specific choice of the distinguished elements $h(k):=x^{\epsilon_k}D_k$, $e(k):=x^{2\epsilon_k}D_k$ with $[h(k),e(k)]=e(k)$ and $h(m):=x^{\epsilon_m}D_m$, $e(m):=x^{2\epsilon_m}D_m$ $(1\le m\ne k\le n)$ with $[h(m),e(m)]=e(m)$, the corresponding quantization integral form of $U(\mathbf{W}_{\mathbb{Z}}^{+})$ over $U(\mathbf{W}_{\mathbb{Z}}^{+})[[t]]$ by the Drinfel'd twist $\mathcal{F}=\mathcal{F}(m)\mathcal{F}(k)$ with the algebra structure undeformed is given by*

$$\begin{aligned}\Delta(x^\alpha D_i)=x^\alpha D_i\otimes(1-e(k)t)^{\alpha_k-\delta_{ik}}(1-e(m)t)^{\alpha_m-\delta_{im}}+\sum_{\ell,n=0}^{\infty}(-1)^{\ell+n}\\ \cdot C(k)_\ell C(m)_n h(k)^{\langle\ell\rangle}h(m)^{\langle n\rangle}\otimes(1-e(k)t)^{-\ell}(1-e(m)t)^{-n}\cdot x^{\alpha+\ell\epsilon_k+n\epsilon_m}D_it^{\ell+n},\end{aligned}\tag{27}$$

$$\begin{aligned}S(x^\alpha D_i)=-(1-e(k)t)^{-\alpha_k+\delta_{ik}}(1-e(m)t)^{-\alpha_m+\delta_{im}}\Big(\sum_{\ell,n=0}^{\infty}C(k)_\ell C(m)_n\\ \cdot x^{\alpha+\ell\epsilon_k+n\epsilon_m}D_i\cdot h(m)_1^{\langle n\rangle}h(k)_1^{\langle\ell\rangle}t^{\ell+n}\Big),\end{aligned}\tag{28}$$

$$\varepsilon(x^\alpha D_i)=0,\tag{29}$$

where $\alpha\in\mathbb{Z}_+^n$, $C(k)_\ell=A(k)_\ell-B(k)_\ell$, $A(k)_\ell$, $B(k)_\ell$ as in Corollary 2.10.

Proof. Using Corollary 2.10, we get

$$\begin{aligned}\Delta(x^\alpha D_i)&=\mathcal{F}(m)\mathcal{F}(k)\Delta_0(x^\alpha D_i)\mathcal{F}(k)^{-1}\mathcal{F}(m)^{-1}\\ &=\mathcal{F}(m)\Big(x^\alpha D_i\otimes(1-e(k)t)^{\alpha_k-\delta_{ik}}\\ &\qquad+\sum_{\ell=0}^{\infty}(-1)^{\ell}C(k)_\ell h(k)^{\langle\ell\rangle}\otimes(1-e(k)t)^{-\ell}\cdot x^{\alpha+\ell\epsilon_k}D_it^{\ell}\Big)\mathcal{F}(m)^{-1}\end{aligned}$$

Using (14) and Lemma 2.2, we get

$$\begin{aligned}
&\mathcal{F}(m)\Big(x^{\alpha}D_i\otimes(1-e(k)t)^{\alpha_k-\delta_{ik}}\Big)\mathcal{F}(m)^{-1}\\
&\quad=\mathcal{F}(m)\Big(x^{\alpha}D_i\otimes 1\Big)\mathcal{F}(m)^{-1}\Big(1\otimes(1-e(k)t)^{\alpha_k-\delta_{ik}}\Big)\\
&\quad=\mathcal{F}(m)\mathcal{F}(m)^{-1}_{\delta_{im}-\alpha_m}\Big(x^{\alpha}D_i\otimes 1\Big)\Big(1\otimes(1-e(k)t)^{\alpha_k-\delta_{ik}}\Big)\\
&\quad=\Big(1\otimes(1-e(m)t)^{\alpha_m-\delta_{im}}\Big)\Big(x^{\alpha}D_i\otimes 1\Big)\Big(1\otimes(1-e(k)t)^{\alpha_k-\delta_{ik}}\Big)\\
&\quad=x^{\alpha}D_i\otimes(1-e(k)t)^{\alpha_k-\delta_{ik}}(1-e(m)t)^{\alpha_m-\delta_{im}}.
\end{aligned}$$

Using (15), we get

$$\begin{aligned}
&\mathcal{F}(m)\Big(\sum_{\ell=0}^{\infty}(-1)^{\ell}C(k)_{\ell}h(k)^{\langle\ell\rangle}\otimes(1-e(k)t)^{-\ell}\cdot x^{\alpha+\ell\epsilon_k}D_it^{\ell}\Big)\mathcal{F}(m)^{-1}\\
&\quad=\sum_{\ell=0}^{\infty}(-1)^{\ell}C(k)_{\ell}(h(k)^{\langle\ell\rangle}\otimes(1-e(k)t)^{-\ell})\cdot\mathcal{F}(m)\Big(1\otimes x^{\alpha+\ell\epsilon_k}D_it^{\ell}\Big)\mathcal{F}(m)^{-1}\\
&\quad=\sum_{\ell,n=0}^{\infty}(-1)^{\ell+n}C(k)_{\ell}(h(k)^{\langle\ell\rangle}\otimes(1-e(k)t)^{-\ell})\\
&\qquad\cdot\mathcal{F}(m)\mathcal{F}(m)^{-1}_{n}\cdot\Big(C(m)_nh(m)^{\langle n\rangle}\otimes x^{\alpha+\ell\epsilon_k+n\epsilon_m}D_it^{n+\ell}\Big)\\
&\quad=\sum_{\ell,n=0}^{\infty}(-1)^{\ell+n}C(k)_{\ell}C(m)_nh(k)^{\langle\ell\rangle}h(m)^{\langle n\rangle}\otimes\\
&\qquad(1-e(k)t)^{-\ell}(1-e(m)t)^{-n}\cdot x^{\alpha+\ell\epsilon_k+n\epsilon_m}D_it^{\ell+n}.
\end{aligned}$$

For $k\neq m$, using the definitions of v and u, we get $v=v(k)v(m)=v(m)v(k)$ and $u=u(m)u(k)=u(k)u(m)$. By Corollary 2.10 and using (16), we have

$$\begin{aligned}
S(x^{\alpha}D_i)&=-v\cdot x^{\alpha}D_i\cdot u=-v(m)v(k)\cdot x^{\alpha}D_i\cdot u(k)u(m)\\
&=v(m)\cdot\Big(-(1-e(k)t)^{-\alpha_k+\delta_{ik}}\cdot\Big(\sum_{\ell=0}^{\infty}C(k)_{\ell}x^{\alpha+\ell\epsilon_k}D_i\cdot h(k)_1^{\langle\ell\rangle}t^{\ell}\Big)\Big)\cdot u(m)\\
&=-(1-e(k)t)^{-\alpha_k+\delta_{ik}}\cdot v(m)u(m)_{\alpha_m-\delta_{im}}\\
&\qquad\cdot\Big(\sum_{\ell,n=0}^{\infty}C(k)_{\ell}C(m)_nx^{\alpha+\ell\epsilon_k+n\epsilon_m}D_i\cdot h(m)_1^{\langle n\rangle}h(k)_1^{\langle\ell\rangle}t^{n+\ell}\Big)\\
&=-(1-e(k)t)^{-\alpha_k+\delta_{ik}}(1-e(m)t)^{-\alpha_m+\delta_{im}}\\
&\qquad\cdot\Big(\sum_{\ell,n=0}^{\infty}C(k)_{\ell}C(m)_nx^{\alpha+\ell\epsilon_k+n\epsilon_m}D_i\cdot h(m)_1^{\langle n\rangle}h(k)_1^{\langle\ell\rangle}t^{n+\ell}\Big).
\end{aligned}$$

Therefore, the proof is complete. □

Let $I=\{1,\cdots,n\}$, $\eta=(\eta_1,\cdots,\eta_n)$ $(\eta_i\in\{0,1\})$, $\ell=(\ell_1,\cdots,\ell_n)\in\mathbb{N}^n$. Then $|\eta\ell|=\sum_i\eta_i\ell_i$. Set $(1-e(I)t)^{\eta(\alpha-\delta_{iI})}=\prod_{k\in I}(1-e(k)t)^{\eta_k(\alpha_k-\delta_{ik})}$ and $h(I)^{\langle\eta\ell\rangle}=\prod_{k\in I}h(k)^{\langle\eta_k\ell_k\rangle}$.

More generally, we can get the following result.

Theorem 2.14. *The corresponding quantization integral form of $U(\mathbf{W}_{\mathbb{Z}}^+)$ over $U(\mathbf{W}_{\mathbb{Z}}^+)[[t]]$ by the Drinfel'd twist $\mathcal{F}^\eta=\mathcal{F}(1)^{\eta_1}\cdots\mathcal{F}(n)^{\eta_n}$ with the algebra structure undeformed is given by*

$$\Delta(x^\alpha D_i)=x^\alpha D_i\otimes(1-e(I)t)^{\eta(\alpha-\delta_{iI})}+\sum_{\eta\ell\geq\underline{0}}(-1)^{|\eta\ell|}h(I)^{\langle\eta\ell\rangle}\otimes C(I)_\ell^\eta(1-e(I)t)^{-\eta\ell}\cdot x^{\alpha+\eta\ell}D_it^{|\eta\ell|},\tag{30}$$

$$S(x^\alpha D_i)=-(1-e(I)t)^{-\eta(\alpha-\delta_{iI})}\Big(\sum_{\eta\ell\geq\underline{0}}C(I)_\ell^\eta x^{\alpha+\eta\ell}D_i\cdot h(I)_1^{\langle\eta\ell\rangle}t^{|\eta\ell|}\Big),\tag{31}$$

$$\varepsilon(x^\alpha D_i)=0,\tag{32}$$

where $\alpha\in\mathbb{Z}_+^n$, $C(I)_\ell=\prod_{k\in I}(A(k)_{\ell_k}-B(k)_{\ell_k})^{\eta_k}$, $A(k)_{\ell_k}$, $B(k)_{\ell_k}$ as in Corollary 2.10.

3. Quantization of the Jacobson-Witt algebra in the modular case

In this section, our main purposes are twofold: Firstly, in view of Lemma 1.2, we make *the modulo p reduction* for the quantization integral form of $U(\mathbf{W}_{\mathbb{Z}}^+)$ in characteristic 0 obtained in Corollary 2.10 to yield the quantization of $U(\mathbf{W}(n;\underline{1}))$, for the Cartan type restricted simple modular Lie algebra $\mathbf{W}(n;\underline{1})$ of W type in characteristic p. Secondly, we shall further make *the "p-restrictedness" reduction* for the quantization of $U(\mathbf{W}(n;\underline{1}))$, which will lead to the required quantization of $\mathbf{u}(\mathbf{W}(n;\underline{1}))$ (here $\mathbf{u}(\mathbf{W}(n;\underline{1}))$ is the restricted universal enveloping algebra of $\mathbf{W}(n;\underline{1})$).

3.1. Modulo p reduction. Let $\mathbb{Z}_p$ be the prime subfield of $\mathcal{K}$ with $\mathrm{char}(\mathcal{K})=p$. When considering $\mathbf{W}_{\mathbb{Z}}^+$ as a $\mathbb{Z}_p$-Lie algebra, namely, making a modulo p reduction for the defining relations of $\mathbf{W}_{\mathbb{Z}}^+$, we denote it by $\mathbf{W}_{\mathbb{Z}_p}^+$. By Lemma 1.2 (2), we see that $(J_{\underline{1}})_{\mathbb{Z}_p}=\mathrm{Span}_{\mathbb{Z}_p}\{x^\alpha D_i\mid\exists j:\alpha_j\geq p\}$ is a maximal ideal of $\mathbf{W}_{\mathbb{Z}_p}^+$, and $\mathbf{W}_{\mathbb{Z}_p}^+/(J_{\underline{1}})_{\mathbb{Z}_p}\cong\mathbf{W}(n;\underline{1})_{\mathbb{Z}_p}=\mathrm{Span}_{\mathbb{Z}_p}\{x^{(\alpha)}D_i\mid 0\leq\alpha\leq\tau,1\leq i\leq n\}$. Moreover, we have $\mathbf{W}(n;\underline{1})=\mathcal{K}\otimes_{\mathbb{Z}_p}\mathbf{W}(n;\underline{1})_{\mathbb{Z}_p}=\mathcal{K}\mathbf{W}(n;\underline{1})_{\mathbb{Z}_p}$, and $\mathbf{W}_{\mathcal{K}}^+=\mathcal{K}\mathbf{W}_{\mathbb{Z}_p}^+$.

Observe that the ideal $J_{\underline{1}}:=\mathcal{K}(J_{\underline{1}})_{\mathbb{Z}_p}$ generates an ideal of $U(\mathbf{W}_{\mathcal{K}}^+)$ over $\mathcal{K}$, denoted by $J:=J_{\underline{1}}U(\mathbf{W}_{\mathcal{K}}^+)$, where $\mathbf{W}_{\mathcal{K}}^+/J_{\underline{1}}\cong\mathbf{W}(n;\underline{1})$. Based on the formulae (24) & (25), we see that J is a Hopf ideal of $U(\mathbf{W}_{\mathcal{K}}^+)$ such that $U(\mathbf{W}_{\mathcal{K}}^+)/J\cong U(\mathbf{W}(n;\underline{1}))$. Note that the elements $\frac{1}{\alpha!}x^\alpha D_i$ in $\mathbf{W}_{\mathcal{K}}^+$ for $0\leq\alpha\leq\tau$ will be identified with $x^{(\alpha)}D_i$ in $\mathbf{W}(n;\underline{1})$ and those in $J_{\underline{1}}$ with 0. Hence, by Corollary 2.10, we get the quantization of $U(\mathbf{W}(n;\underline{1}))$ over $U(\mathbf{W}(n;\underline{1}))[[t]]$ as follows.

Theorem 3.1. *Given two distinguished elements $h:=x^{(\epsilon_k)}D_k$, $e:=2x^{(2\epsilon_k)}D_k$ $(1\leq k\leq n)$ with $[h,e]=e$, the corresponding quantization of $U(\mathbf{W}(n;\underline{1}))$ over $U(\mathbf{W}(n;\underline{1}))[[t]]$ with the algebra structure undeformed is given by*

$$\Delta(x^{(\alpha)}D_i)=x^{(\alpha)}D_i\otimes(1-et)^{\alpha_k-\delta_{ik}}+\sum_{\ell=0}^{p-1}(-1)^\ell\bar{C}_\ell\,h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}x^{(\alpha+\ell\epsilon_k)}D_it^\ell,\tag{33}$$

$$S(x^{(\alpha)}D_i)=-(1-et)^{-\alpha_k+\delta_{ik}}\cdot\Big(\sum_{\ell=0}^{p-1}\bar{C}_\ell\, x^{(\alpha+\ell\epsilon_k)}D_i\cdot h_1^{\langle\ell\rangle}t^\ell\Big), \tag{34}$$

$$\varepsilon(x^{(\alpha)}D_i)=0, \tag{35}$$

where $0\le\alpha\le\tau$, $\bar{C}_\ell=\bar{A}_\ell-\bar{B}_\ell$, $\bar{A}_\ell=\ell!\binom{\alpha_k+\ell}{\ell}A_\ell\,(\mathrm{mod}\,p)$, $\bar{B}_\ell=\ell!\binom{\alpha_k+\ell}{\ell}B_\ell\,(\mathrm{mod}\,p)$.

3.2. Modulo "p-restrictedness" reduction. Assume that I is the ideal of $U(\mathbf{W}(n;\underline{1}))$ over $\mathcal{K}$ generated by $(x^{(\epsilon_i)}D_i)^p-x^{(\epsilon_i)}D_i$ and $(x^{(\alpha)}D_i)^p$ with $\alpha\ne\epsilon_i$ for $0\le\alpha\le\tau$ and $1\le i\le n$. $\mathbf{u}(\mathbf{W}(n;\underline{1}))=U(\mathbf{W}(n;\underline{1}))/I$ is of dimension p^{np^n}. In order to get a reasonable quantization of finite dimension for $\mathbf{u}(\mathbf{W}(n;\underline{1}))$ in characteristic p, at first, it is necessary to clarify (in concept) what is the underlying vector space in which the required t-deformed object exists. According to our modulo p reduction approach, it should be induced from the topologically free $\mathcal{K}[[t]]$-algebra $U(\mathbf{W}(n;\underline{1}))[[t]]$ given in Theorem 3.1, or more naturally, related with the topologically free $\mathcal{K}[[t]]$-algebra $\mathbf{u}(\mathbf{W}(n;\underline{1}))[[t]]$. As $\mathbf{u}(\mathbf{W}(n;\underline{1}))$ is finite dimensional, as algebras, we have (see p. 2, [**4**])

$$\mathbf{u}(\mathbf{W}(n;\underline{1}))[[t]]\cong\mathbf{u}(\mathbf{W}(n;\underline{1}))\otimes_{\mathcal{K}}\mathcal{K}[[t]]. \tag{36}$$

So the standard Hopf algebra structure $(\mathbf{u}(\mathbf{W}(n;\underline{1}))[[t]],m,\iota,\Delta_0,S_0,\varepsilon_0)$ can be viewed as the tensor Hopf algebra structure of the standard ones on $\mathbf{u}(\mathbf{W}(n;\underline{1}))$ and on $\mathcal{K}[[t]]$, while the expected *twisted* Hopf algebra structure on $(\mathbf{u}(\mathbf{W}(n;\underline{1}))[[t]],m,\iota,\Delta,S,\varepsilon)$ induced from Theorem 3.1 when restricted to the sub-Hopf algebra $\mathcal{K}[[t]]$ should still go back to the original one on $\mathcal{K}[[t]]$ itself.

Keep the above viewpoint in mind. Furthermore, we need another observation below.

LEMMA 3.2. (i) $(1-et)^p\equiv 1\pmod{p,I}$.
(ii) $(1-et)^{-1}\equiv 1+et+\cdots+e^{p-1}t^{p-1}\pmod{p,I}$.
(iii) $h_a^{\langle\ell\rangle}\equiv 0\pmod{p,I}$ *for* $\ell\ge p$, *and* $a\in\mathbb{Z}_p$.

PROOF. (i), (ii) follow from $e^p=0$ in $\mathbf{u}(\mathbf{W}(n;\underline{1}))$.

(iii) For $\ell\in\mathbb{Z}_+$, there is a unique decomposition $\ell=\ell_0+\ell_1p$ with $0\le\ell_0<p$ and $\ell_1\ge0$. Using the formulae (1) & (3), we have

$$\begin{aligned}h_a^{\langle\ell\rangle}&=h_a^{\langle\ell_0\rangle}\cdot h_{a+\ell_0}^{\langle\ell_1p\rangle}\equiv h_a^{\langle\ell_0\rangle}\cdot(h_{a+\ell_0}^{\langle p\rangle})^{\ell_1}\qquad(\mathrm{mod}\ p)\\&\equiv h_a^{\langle\ell_0\rangle}\cdot(h^p-h)^{\ell_1}\quad(\mathrm{mod}\ p),\end{aligned}$$

where we used the facts that $(x+1)(x+2)\cdots(x+p-1)\equiv x^{p-1}-1$ (mod p), and $(x+a+\ell_0)^p\equiv x^p+a+\ell_0$ (mod p). Hence, $h_a^{\langle\ell\rangle}\equiv0$ (mod p, I) for $\ell\ge p$. □

This Lemma, combining with Theorem 3.1, indicates that the required t-deformation of $\mathbf{u}(\mathbf{W}(n;\underline{1}))$ (if it exists) only happens in a p-truncated polynomial ring (with degrees of t less than p) with coefficients in $\mathbf{u}(\mathbf{W}(n;\underline{1}))$. Let $\mathcal{K}[t]_p$ be a p-truncated polynomial ring (of small possible dimension) over $\mathcal{K}$, which should inherit a standard Hopf algebra structure from that on $\mathcal{K}[[t]]$ with respect to modulo a Hopf ideal of it. In the modular case, such a Hopf ideal in $\mathcal{K}[[t]]$ has to take the form (t^p-qt) generated by a p-polynomial t^p-qt of degree p for a parameter $q\in\mathcal{K}$. Denote by $\mathcal{K}[t]_p^{(q)}$ the corresponding quotient ring. That is to say

$$\mathcal{K}[t]_p^{(q)}\cong\mathcal{K}[[t]]/(t^p-qt),\qquad \textit{for}\ \ q\in\mathcal{K}. \tag{37}$$

Thereby, we obtain the underlying ring for our required t-deformation of $\mathbf{u}(\mathbf{W}(n;\underline{1}))$ over $\mathcal{K}[t]_p^{(q)}$, denoted by $\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1}))$. Moreover, as standard Hopf algebras,

$$\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1})) \cong \mathbf{u}(\mathbf{W}(n;\underline{1})) \otimes_{\mathcal{K}} \mathcal{K}[t]_p^{(q)}. \tag{38}$$

Hence, $\dim_{\mathcal{K}} \mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1})) = p \cdot \dim_{\mathcal{K}} \mathbf{u}(\mathbf{W}(n;\underline{1})) = p^{1+np^n}$.

Definition 3.3. With notations as above. A Hopf algebra $(\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1})), m, \iota, \Delta, S, \varepsilon)$ is said to be a quantization of $\mathbf{u}(\mathbf{W}(n;\underline{1}))$ (in characteristic p) if it is a twisting of the standard Hopf algebra structure (as a tensor Hopf algebra of the standard ones on $\mathbf{u}(\mathbf{W}(n;\underline{1}))$ and on $\mathcal{K}[t]_p^{(q)}$) with $\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1}))/t\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1})) \cong \mathbf{u}(\mathbf{W}(n;\underline{1}))$.

To describe $\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1}))$ explicitly, we still need an auxiliary Lemma.

Lemma 3.4. (i) *$d^{(\ell)}(x^{(\alpha)}D_i) = \bar{C}_\ell x^{(\alpha+\ell\epsilon_k)}D_i$, where $d^{(\ell)} = \frac{1}{\ell!}(\mathrm{ad}\, e)^\ell$, $e = 2x^{(2\epsilon_k)}D_k$, and $\bar{C}_\ell$ as in Theorem 3.1.*

(ii) *$d^{(\ell)}(x^{(\epsilon_i)}D_i) = \delta_{\ell,0}x^{(\epsilon_i)}D_i - \delta_{1,\ell}\delta_{ik}e$.*

(iii) *$d^{(\ell)}((x^{(\alpha)}D_i)^p) = \delta_{\ell,0}(x^{(\alpha)}D_i)^p - \delta_{1,\ell}\delta_{ik}\delta_{\alpha,\epsilon_i}e$.*

Proof. (i) For $0 \le \alpha \le \tau$, by (19) and Theorem 3.1 (noting $B_\ell = \delta_{ik}A_{\ell-1}$), we get

$$\begin{aligned} d^{(\ell)}(x^{(\alpha)}D_i) &= \frac{1}{\alpha!} d^{(\ell)}(x^{\alpha-\epsilon_i}\partial_i) = \frac{1}{\alpha!} x^{\alpha-\epsilon_i+\ell\epsilon_k}(A_\ell \partial_i - B_\ell \partial_k) \\ &= \ell! \binom{\alpha_k+\ell}{\ell} C_\ell\, x^{(\alpha+\ell\epsilon_k)}D_i \\ &= \bar{C}_\ell\, x^{(\alpha+\ell\epsilon_k)}D_i. \end{aligned}$$

(ii) Noting that

$$\bar{A}_\ell = \ell! \binom{\alpha_k+\ell}{\ell} A_\ell = \binom{\alpha_k+\ell}{\ell} \prod_{j=0}^{\ell-1}(\alpha_k - \delta_{ik} + j), \qquad \bar{B}_\ell = \ell! \binom{\alpha_k+\ell}{\ell} \delta_{ik}A_{\ell-1},$$

we obtain the following

If $i = k$: $\bar{A}_0 = 1$, $\bar{A}_\ell = 0$ for $\ell \ge 1$ and, $\bar{B}_0 = 0$, $\bar{B}_1 = 2$, $\bar{B}_\ell = 0$ for $\ell \ge 2$. So by Lemma 3.4 (i), we have $d^{(1)}(x^{(\epsilon_k)}D_k) = -2x^{(2\epsilon_k)}D_k = -e$, and $d^{(\ell)}(x^{(\epsilon_k)}D_k) = 0$ for $\ell \ge 2$.

If $i \ne k$: $\bar{A}_0 = 1$, $\bar{A}_\ell = 0$ for $\ell \ge 1$, and $\bar{B}_\ell = 0$ for $\ell \ge 0$, namely, $d^{(\ell)}(x^{(\epsilon_i)}D_i) = 0$ for $\ell \ge 1$.

In both cases, we arrive at the result as required.

(iii) From (12), we obtain that for $0 \le \alpha \le \tau$,

$$\begin{aligned} d^{(1)}\left((x^{(\alpha)}D_i)^p\right) &= \frac{1}{(\alpha!)^p}[\, e, (x^\alpha D_i)^p\,] = \frac{1}{\alpha!}[\, e, (x^{\alpha-\epsilon_i}\partial_i)^p\,] \\ &= \frac{1}{\alpha!}\sum_{\ell=1}^{p}(-1)^\ell \binom{p}{\ell} (x^{\alpha-\epsilon_i}\partial_i)^{p-\ell} \cdot x^{\epsilon_k+\ell(\alpha-\epsilon_i)}(a_\ell\partial_k - b_\ell\partial_i) \\ &\equiv -\frac{a_p}{\alpha!}\, x^{2\epsilon_k+p(\alpha-\epsilon_i)}D_k \qquad (\mathrm{mod}\ \ p\,) \\ &\equiv \begin{cases} -a_p\, e, & \textit{if} \quad \alpha = \epsilon_i \\ 0, & \textit{if} \quad \alpha \ne \epsilon_i \end{cases} \qquad (\mathrm{mod}\ \ J), \end{aligned}$$

where we get the last "$\equiv$" by using the identification with respect to modulo the ideal J as before, and $a_\ell = \prod_{j=0}^{\ell-1}(\delta_{ik}+j(\alpha_i-1))$, $b_\ell = \ell\,(\alpha_k-\delta_{ik})a_{\ell-1}$, and $a_p=\delta_{ik}$ for $\alpha=\epsilon_i$.

Consequently, by definition of $d^{(\ell)}$, we obtain $d^{(\ell)}((x^{(\alpha)}D_i)^p)=0$ in $\mathbf{u}(\mathbf{W}(n;\underline{1}))$ for $2\le \ell\le p-1$ and any $0\le\alpha\le\tau$. □

Based on Theorem 3.1, Definition 3.3 and Lemma 3.4, we arrive at

THEOREM 3.5. *With the specific choice of the two distinguished elements $h := x^{(\epsilon_k)}D_k$, $e := 2x^{(2\epsilon_k)}D_k$ $(1\le k\le n)$ with $[h,e]=e$, there is a noncommutative and noncocummtative Hopf algebra $(\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1})),m,\iota,\Delta,S,\varepsilon)$ over $\mathcal{K}[t]_p^{(q)}$ with its algebra structure undeformed, whose coalgebra structure is given by*

(39)

$$\Delta(x^{(\alpha)}D_i)=x^{(\alpha)}D_i\otimes(1-et)^{\alpha_k-\delta_{ik}}+\sum_{\ell=0}^{p-1}(-1)^\ell h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}\cdot d^{(\ell)}(x^{(\alpha)}D_i)t^\ell,$$

$$S(x^{(\alpha)}D_i)=-(1-et)^{\delta_{ik}-\alpha_k}\Big(\sum_{\ell=0}^{p-1}d^{(\ell)}(x^{(\alpha)}D_i)\cdot h_1^{\langle\ell\rangle}t^\ell\Big), \tag{40}$$

$$\varepsilon(x^{(\alpha)}D_i)=0, \tag{41}$$

where $0\le\alpha\le\tau$. It is finite dimensional and $\dim_{\mathcal{K}}\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1}))=p^{1+np^n}$.

PROOF. Let I_t be the ideal of $(U(\mathbf{W}(n;\underline{1}))[[t]],m,\iota,\Delta,S,\varepsilon)$ generated by I and t^p-qt $(q\in\mathcal{K})$. In what follows, we shall show that the ideal I_t is a Hopf ideal of the *twisted* Hopf algebra $U(\mathbf{W}(n;\underline{1}))[[t]]$ given in Theorem 3.1. To this end, it suffices to verify that Δ and S preserve the elements in I since $\Delta(t^p-qt) = (t^p-qt)\otimes 1+1\otimes(t^p-qt)$ and $S(t^p-qt)=-(t^p-qt)$.

(I) By Lemmas 2.8, 3.2 & 3.4 (iii), we obtain

$$\begin{aligned}
\Delta((x^{(\alpha)}D_i)^p) &= (x^{(\alpha)}D_i)^p\otimes(1-et)^{p(\alpha_k-\delta_{ik})}\\
&\quad+\sum_{\ell=0}^{\infty}(-1)^\ell h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}d^{(\ell)}((x^{(\alpha)}D_i)^p)t^\ell\\
&\equiv (x^{(\alpha)}D_i)^p\otimes 1+\sum_{\ell=0}^{p-1}(-1)^\ell h^{\langle\ell\rangle}\otimes(1-et)^{-\ell}d^{(\ell)}((x^{(\alpha)}D_i)^p)t^\ell\\
&\qquad(\mathrm{mod}\ \ p,\ I_t\otimes U(\mathbf{W}(n;\underline{1}))[[t]]+U(\mathbf{W}(n;\underline{1}))[[t]]\otimes I_t)\\
&= (x^{(\alpha)}D_i)^p\otimes 1+1\otimes(x^{(\alpha)}D_i)^p+h\otimes(1-et)^{-1}\delta_{ik}\delta_{\alpha,\epsilon_i}et.\\
&\qquad(\mathrm{mod}\ \ I_t\otimes U(\mathbf{W}(n;\underline{1}))[[t]]+U(\mathbf{W}(n;\underline{1}))[[t]]\otimes I_t)
\end{aligned} \tag{42}$$

Hence, when $\alpha\ne\epsilon_i$, we get

$$\begin{aligned}
\Delta((x^{(\alpha)}D_i)^p)&\equiv(x^{(\alpha)}D_i)^p\otimes 1+1\otimes(x^{(\alpha)}D_i)^p\\
&\subseteq I_t\otimes U(\mathbf{W}(n;\underline{1}))[[t]]+U(\mathbf{W}(n;\underline{1}))[[t]]\otimes I_t.
\end{aligned}$$

When $\alpha=\epsilon_i$, by Lemma 3.4 (ii), (27) becomes

$$\Delta(x^{(\epsilon_i)}D_i)=x^{(\epsilon_i)}D_i\otimes 1+1\otimes x^{(\epsilon_i)}D_i+\delta_{ik}h\otimes(1-et)^{-1}et.$$

Combining with (42), we obtain

$$\begin{aligned}\Delta((x^{(\epsilon_i)}D_i)^p - x^{(\epsilon_i)}D_i) &\equiv ((x^{(\epsilon_i)}D_i)^p - x^{(\epsilon_i)}D_i)\otimes 1 + 1\otimes((x^{(\epsilon_i)}D_i)^p - x^{(\epsilon_i)}D_i)\\ &\subseteq I_t\otimes U(\mathbf{W}(n;\underline{1}))[[t]] + U(\mathbf{W}(n;\underline{1}))[[t]]\otimes I_t.\end{aligned}$$

Thereby, we prove that the ideal I_t is also a coideal of the Hopf algebra $U(\mathbf{W}(n;\underline{1}))[[t]]$.

(II) By Lemmas 2.8, 3.2 & 3.4 (iii), we have

$$\begin{aligned}S((x^{(\alpha)}D_i)^p) &= -(1-et)^{-p(\alpha_k-\delta_{ik})}\sum_{\ell=0}^{\infty} d^{(\ell)}((x^{(\alpha)}D_i)^p)\cdot h_1^{\langle\ell\rangle}t^\ell\\ &\equiv -(x^{(\alpha)}D_i)^p - \sum_{\ell=1}^{p-1} d^{(\ell)}((x^{(\alpha)}D_i)^p)\cdot h_1^{\langle\ell\rangle}t^\ell \pmod{(p,I)}\\ &= -(x^{(\alpha)}D_i)^p + \delta_{ik}\delta_{\alpha,\epsilon_i}e\cdot h_1^{\langle 1\rangle}t.\end{aligned}\tag{43}$$

Hence, when $\alpha\neq\epsilon_i$, we get

$$S\big((x^{(\alpha)}D_i)^p\big)\equiv -(x^{(\alpha)}D_i)^p\equiv 0 \pmod{I_t}.$$

When $\alpha=\epsilon_i$, by Lemma 3.4 (ii), (28) reads as $S(x^{(\epsilon_i)}D_i) = -x^{(\epsilon_i)}D_i + \delta_{ik}e\cdot h_1^{\langle 1\rangle}t$. Combining with (43), we obtain

$$S\big((x^{(\epsilon_i)}D_i)^p - x^{(\epsilon_i)}D_i\big)\equiv -\big((x^{(\epsilon_i)}D_i)^p - x^{(\epsilon_i)}D_i\big)\equiv 0 \pmod{I_t}.$$

Thereby, we show that the ideal I_t is indeed preserved by the antipode S of the quantization $U(\mathbf{W}(n;\underline{1}))[[t]]$ given in Theorem 3.1.

(III) It is obvious to notice that $\varepsilon((x^{(\alpha)}D_i)^p)=0$ for all $0\le\alpha\le\tau$.

In other words, we prove that I_t is a Hopf ideal in $U(\mathbf{W}(n;\underline{1}))[[t]]$. We thus obtain the required t-deformation on $\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1}))$ for the Cartan type simple modular restricted Lie algebra of W type — the Jacobson-Witt algebra $\mathbf{W}(n;\underline{1})$. □

Remark 3.6. (i) If we set $f=(1-et)^{-1}$, then by Lemma 3.4, Theorem 3.5, we have

$$[h,f]=f^2-f,\quad h^p=h,\quad f^p=1,\quad \Delta(h)=h\otimes f+1\otimes h,$$

where f is a group-like element, and $S(h)=-hf^{-1}$, $\varepsilon(h)=0$. So the subalgebra generated by h and f is a Hopf subalgebra of $\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1}))$, which is isomorphic to the well-known Radford Hopf algebra over $\mathcal{K}$ in characteristic p.

(ii) According to our argument above, given a parameter $q\in\mathcal{K}$, one can specialize t to any one of roots of the p-polynomial $t^p-qt\in\mathcal{K}[t]$ in a split field of $\mathcal{K}$. For instance, if take $q=1$, then one can specialize t to any scalar in $\mathbb{Z}_p$. If set $t=0$, then we get the original standard Hopf algebra structure of $\mathbf{u}(\mathbf{W}(n;\underline{1}))$. In this way, we indeed get a new Hopf algebra structure over the same restricted universal enveloping algebra $\mathbf{u}(\mathbf{W}(n;\underline{1}))$ over $\mathcal{K}$ under the assumption that $\mathcal{K}$ is algebraically closed, which has the new coalgebra structure induced by Theorem 3.5, but has dimension p^{np^n}.

3.3. More quantizations. Carrying out the modular reduction process for those pairwise different products of some basic Drinfel'd twists as stated in Remark 2.11, we will get many more new families of noncommutative and noncocommutative Hopf algebras of dimension p^{1+np^n} with indeterminate t or of dimension p^{np^n} with specializing t into a scalar in $\mathcal{K}$. We have the following general results about the quantizations under concern.

Maintain the notations as in Theorem 2.14 and set $d_I^{(\eta\ell)} = \prod\limits_{k\in I} d_k^{(\eta_k\ell_k)}$, where $d_k^{(\ell_k)} = \frac{1}{\ell_k!}(\mathrm{ad}\, e(k))^{\ell_k}$. Then we have

Theorem 3.7. *For each given $\eta=(\eta_1,\cdots,\eta_n)$ with $\eta_i\in\{0,1\}$, there exists a noncommutative and noncocummtative Hopf algebra $(\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1})), m, \iota, \Delta, S, \varepsilon)$ over $\mathcal{K}[t]_p^{(q)}$ with the algebra structure undeformed, whose coalgebra structure is given by*

$$\Delta(x^{(\alpha)}D_i) = x^{(\alpha)}D_i \otimes (1-e(I)t)^{\eta(\alpha-\delta_{iI})} + \sum_{\eta\ell=\underline{0}}^{\eta\tau} (-1)^{|\eta\ell|} h(I)^{\langle\eta\ell\rangle} \otimes (1-e(I)t)^{-\eta\ell}\cdot d_I^{(\eta\ell)}(x^{(\alpha)}D_i)t^{|\eta\ell|}, \tag{44}$$

$$S(x^{(\alpha)}D_i) = -(1-e(I)t)^{-\eta(\alpha-\delta_{iI})}\Big(\sum_{\eta\ell=\underline{0}}^{\eta\tau} d_I^{(\eta\ell)}(x^{(\alpha)}D_i)h(I)_1^{\langle\eta\ell\rangle}t^{|\eta\ell|}\Big), \tag{45}$$

$$\varepsilon(x^{(\alpha)}D_i) = 0, \tag{46}$$

where $0\le\alpha\le\tau$. It is finite dimensional and $\dim_{\mathcal{K}} \mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1})) = p^{1+np^n}$.

For the proof, it suffices to work with the case in Lemma 2.13. To this end, we begin to perform the modular reduction process for the quantization integral form corresponding to the Drinfel'd twist $\mathcal{F}=\mathcal{F}(m)\mathcal{F}(k)$ $(m\ne k)$.

Lemma 3.8. *Given two pairs of distinguished elements $h(k):=x^{(\epsilon_k)}D_k$, $e(k):=2x^{(2\epsilon_k)}D_k$ with $[h(k), e(k)]=e(k)$ and $h(m):=x^{(\epsilon_m)}D_m$, $e(m):=2x^{(2\epsilon_m)}D_m$ $(1\le m\ne k\le n)$ with $[h(m), e(m)]=e(m)$, the corresponding quantization of $U(\mathbf{W}(n;\underline{1}))$ over $U(\mathbf{W}(n;\underline{1}))[[t]]$ with the algebra structure undeformed is given by*

$$\Delta(x^{(\alpha)}D_i) = x^{(\alpha)}D_i\otimes(1-e(k)t)^{\alpha_k-\delta_{ik}}(1-e(m)t)^{\alpha_m-\delta_{im}} + \sum_{\ell,n=0}^{p-1}(-1)^{\ell+n} \cdot\bar{C}(k)_\ell\bar{C}(m)_n h(k)^{\langle\ell\rangle}h(m)^{\langle n\rangle}\otimes(1-e(k)t)^{-\ell}(1-e(m)t)^{-n}\cdot x^{(\alpha+\ell\epsilon_k+n\epsilon_m)}D_i t^{\ell+n}, \tag{47}$$

$$S(x^{(\alpha)}D_i) = -(1-e(k)t)^{-\alpha_k+\delta_{ik}}(1-e(m)t)^{-\alpha_m+\delta_{im}}\Big(\sum_{\ell,n=0}^{p-1}\bar{C}(k)_\ell\bar{C}(m)_n \cdot x^{(\alpha+\ell\epsilon_k+n\epsilon_m)}D_i\cdot h(m)_1^{\langle n\rangle}h(k)_1^{\langle\ell\rangle}t^{\ell+n}\Big), \tag{48}$$

$$\varepsilon(x^{(\alpha)}D_i) = 0, \tag{49}$$

where $0\le\alpha\le\tau$, $\bar{C}(k)_\ell=\bar{A}(k)_\ell-\bar{B}(k)_\ell$, $\bar{A}(k)_\ell=\ell!\binom{\alpha_k+\ell}{\ell}A(k)_\ell \,(\mathrm{mod}\,p)$, $\bar{B}(k)_\ell=\ell!\binom{\alpha_k+\ell}{\ell}B(k)_\ell\,(\mathrm{mod}\,p)$.

We have the following two Lemmas about the quantization of $U(\mathbf{W}(n;\underline{1}))$ over $U(\mathbf{W}(n;\underline{1}))[[t]]$ defined in Lemma 3.8.

LEMMA 3.9. *For $s \geq 1$, one has*

$$\text{(i)} \qquad \Delta((x^\alpha D_i)^s) = \sum_{\substack{0\le j\le s\\ n,\ell\ge 0}} \binom{s}{j} (-1)^{n+\ell} (x^\alpha D_i)^j h(k)^{\langle \ell\rangle} h(m)^{\langle n\rangle} \otimes$$

$$(1-e(k)t)^{j(\alpha_k-\delta_{ik})-\ell}(1-e(m)t)^{j(\alpha_m-\delta_{im})-n} d_m^{(n)} d_k^{(\ell)}((x^\alpha D_i)^{s-j})t^{\ell+n}.$$

$$\text{(ii)} \qquad S((x^\alpha D_i)^s) = (-1)^s (1-e(k)t)^{-s(\alpha_k-\delta_{ik})}(1-e(m)t)^{-s(\alpha_m-\delta_{im})}$$

$$\cdot\Big(\sum_{n,\ell=0}^{\infty} d_m^{(n)} d_k^{(\ell)}((x^\alpha D_i)^s)\cdot h(k)_1^{\langle \ell\rangle} h(m)_1^{\langle n\rangle} t^{n+\ell}\Big).$$

PROOF. By Lemma 2.8, (18), (14) and Lemma 2.2, we obtain

$$\begin{aligned}
\Delta((x^\alpha D_i)^s) &= \mathcal{F}\Big(x^\alpha D_i\otimes 1 + 1\otimes x^\alpha D_i\Big)^s \mathcal{F}^{-1}\\
&= \mathcal{F}(m)\Big(\sum_{\substack{0\le j\le s\\ \ell\ge 0}} \binom{s}{j}(-1)^\ell (x^\alpha D_i)^j h(k)^{\langle\ell\rangle}\otimes (1-e(k)t)^{j(\alpha_k-\delta_{ik})-\ell}\\
&\quad \cdot d_k^{(\ell)}((x^\alpha D_i)^{s-j})t^\ell\Big)\mathcal{F}(m)^{-1}\\
&= \mathcal{F}(m)\Big(\sum_{\substack{0\le j\le s\\ \ell\ge 0}} \binom{s}{j}(-1)^\ell ((x^\alpha D_i)^j\otimes 1)(h(k)^{\langle\ell\rangle}\otimes (1-e(k)t)^{j(\alpha_k-\delta_{ik})-\ell})\\
&\quad \cdot (1\otimes d_k^{(\ell)}((x^\alpha D_i)^{s-j})t^\ell)\Big)\mathcal{F}(m)^{-1}\\
&= \mathcal{F}(m)\sum_{\substack{0\le j\le s\\ n,\ell\ge 0}} \binom{s}{j}(-1)^{n+\ell}((x^\alpha D_i)^j\otimes 1)\mathcal{F}(m)_n^{-1} h(k)^{\langle\ell\rangle}h(m)^{\langle n\rangle}\\
&\quad \otimes (1-e(k)t)^{j(\alpha_k-\delta_{ik})-\ell} d_m^{(n)} d_k^{(\ell)}((x^\alpha D_i)^{s-j})t^{\ell+n}\\
&= \sum_{\substack{0\le j\le s\\ n,\ell\ge 0}} \binom{s}{j}(-1)^{n+\ell}\mathcal{F}(m)\mathcal{F}(m)^{-1}_{n-j(\alpha_m-\delta_{im})}((x^\alpha D_i)^j\otimes 1)h(k)^{\langle\ell\rangle}h(m)^{\langle n\rangle}\\
&\quad \otimes (1-e(k)t)^{j(\alpha_k-\delta_{ik})-\ell} d_m^{(n)} d_k^{(\ell)}((x^\alpha D_i)^{s-j})t^{\ell+n}\\
&= \sum_{\substack{0\le j\le s\\ n,\ell\ge 0}} \binom{s}{j}(-1)^{n+\ell}(x^\alpha D_i)^j h(k)^{\langle\ell\rangle}h(m)^{\langle n\rangle}\otimes (1-e(k)t)^{j(\alpha_k-\delta_{ik})-\ell}\\
&\quad \cdot (1-e(m)t)^{j(\alpha_m-\delta_{im})-n} d_m^{(n)} d_k^{(\ell)}((x^\alpha D_i)^{s-j})t^{\ell+n}.
\end{aligned}$$

Again by (17) and Lemma 2.2,

$$\begin{aligned}
S((x^\alpha D_i)^s) &= u^{-1}S_0((x^\alpha\partial)^s)\,u = (-1)^s v\cdot (x^\alpha\partial)^s\cdot u\\
&= (-1)^s v(m)\Big((1-e(k)t)^{-s(\alpha_k-\delta_{ik})}\cdot\Big(\sum_{\ell=0}^{\infty} d_k^{(\ell)}((x^\alpha D_i)^s)\cdot h(k)_1^{\langle\ell\rangle}t^\ell\Big)\Big)u(m)\\
&= (-1)^s v(m)u(m)_{s(\alpha_m-\delta_{im})}(1-e(k)t)^{-s(\alpha_k-\delta_{ik})}\\
&\quad \cdot\Big(\sum_{n,\ell=0}^{\infty} d_m^{(n)} d_k^{(\ell)}((x^\alpha D_i)^s)\cdot h(k)_1^{\langle\ell\rangle}h(m)_1^{\langle n\rangle}t^{n+\ell}\Big)
\end{aligned}$$

$$= (-1)^s(1-e(m)t)^{-s(\alpha_m-\delta_{im})}(1-e(k)t)^{-s(\alpha_k-\delta_{ik})}$$
$$\cdot\Big(\sum_{n,\ell=0}^{\infty} d_m^{(n)}d_k^{(\ell)}((x^\alpha D_i)^s)\cdot h(k)_1^{\langle\ell\rangle}h(m)_1^{\langle n\rangle}t^{n+\ell}\Big).$$

So, the proof is complete. □

Lemma 3.4 implies the following

LEMMA 3.10. (i) $d_m^{(n)}d_k^{(\ell)}(x^{(\alpha)}D_i) = \bar{C}(k)_\ell\bar{C}(m)_n x^{(\alpha+\ell\epsilon_k+n\epsilon_m)}D_i$, *where* $d_k^{(\ell)} = \frac{1}{\ell!}(\mathrm{ad}\, e(k))^\ell$, $e(k) = 2x^{(2\epsilon_k)}D_k$, *and* $\bar{C}(k)_\ell$, $\bar{C}(m)_n$ *as in Lemma 3.8.*

(ii) $d_m^{(n)}d_k^{(\ell)}(x^{(\epsilon_i)}D_i) = \delta_{\ell,0}\delta_{n,0}x^{(\epsilon_i)}D_i - (\delta_{n,0}\delta_{1,\ell}\delta_{ik}+\delta_{\ell,0}\delta_{1,n}\delta_{im})\,e(i)$.

(iii) $d_m^{(n)}d_k^{(\ell)}((x^{(\alpha)}D_i)^p) = \delta_{\ell,0}\delta_{n,0}(x^{(\alpha)}D_i)^p - (\delta_{n,0}\delta_{1,\ell}\delta_{ik}+\delta_{\ell,0}\delta_{1,n}\delta_{im})\,\delta_{\alpha,\epsilon_i}e(i)$.

Using Lemmas 3.2, 3.4, 3.9 & 3.10, we get a new Hopf algebra structure over the same algebra $\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1}))$ as follows.

THEOREM 3.11. *Given two pairs of distinguished elements* $h(k) := x^{(\epsilon_k)}D_k$, $e(k) := 2x^{(2\epsilon_k)}D_k$ *with* $[h(k), e(k)] = e(k)$; $h(m) := x^{(\epsilon_m)}D_m$, $e(m) := 2x^{(2\epsilon_m)}D_m$ $(1 \le m \ne k \le n)$ *with* $[h(m), e(m)] = e(m)$, *there is a noncommutative and noncocummtative Hopf algebra* $(\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1})), m, \iota, \Delta, S, \varepsilon)$ *over* $\mathcal{K}[t]_p^{(q)}$ *with its algebra structure undeformed, whose coalgebra structure is given by*

$$\Delta(x^{(\alpha)}D_i) = x^{(\alpha)}D_i\otimes(1-e(k)t)^{\alpha_k-\delta_{ik}}(1-e(m)t)^{\alpha_m-\delta_{im}} + \sum_{n,\ell=0}^{p-1}(-1)^{\ell+n} \tag{50}$$
$$\cdot h(k)^{\langle\ell\rangle}h(m)^{\langle n\rangle}\otimes(1-e(k)t)^{-\ell}(1-e(m)t)^{-n}d_k^{(\ell)}d_m^{(n)}(x^{(\alpha)}D_i)t^{\ell+n},$$
$$S(x^{(\alpha)}D_i) = -(1-e(k)t)^{\delta_{ik}-\alpha_k}(1-e(m)t)^{\delta_{im}-\alpha_m} \tag{51}$$
$$\cdot\Big(\sum_{n,\ell=0}^{p-1} d_k^{(\ell)}d_m^{(n)}(x^{(\alpha)}D_i)h(k)_1^{\langle\ell\rangle}h(m)_1^{\langle n\rangle}t^{\ell+n}\Big),$$
$$\varepsilon(x^{(\alpha)}D_i) = 0, \tag{52}$$

where $0 \le \alpha \le \tau$. *It is finite dimensional and* $\dim_{\mathcal{K}}\mathbf{u}_{t,q}(\mathbf{W}(n;\underline{1})) = p^{1+np^n}$.

PROOF. Let I_t be the ideal of $(U(\mathbf{W}(n;\underline{1}))[[t]], m, \iota, \Delta, S, \varepsilon)$ generated by I and $t^p - qt$ $(q \in \mathcal{K})$. We begin to show that the ideal I_t is a Hopf ideal of the *twisted* Hopf algebra $U(\mathbf{W}(n;\underline{1}))[[t]]$ given in Lemma 3.8. To this purpose, it suffices to verify that Δ and S preserve the elements in I since $\Delta(t^p-qt) = (t^p-qt)\otimes 1 + 1\otimes(t^p-qt)$ and $S(t^p-qt) = -(t^p-qt)$.

(I) By Lemmas 3.9, 3.2, 3.4 & 3.10, we obtain

$$\Delta((x^{(\alpha)}D_i)^p) = (x^{(\alpha)}D_i)^p\otimes(1-e(k)t)^{p(\alpha_k-\delta_{ik})}(1-e(m)t)^{p(\alpha_m-\delta_{im})}$$
$$+\sum_{n,\ell=0}^{\infty}(-1)^{n+\ell}h(k)^{\langle\ell\rangle}h(m)^{\langle n\rangle}\otimes(1-e(k)t)^{-\ell}(1-e(m)t)^{-n} \tag{53}$$
$$\cdot\, d_m^{(n)}d_k^{(\ell)}((x^{(\alpha)}D_i)^p)t^{n+\ell}$$

$$\begin{aligned}
&\equiv (x^{(\alpha)}D_i)^p\otimes 1 + \sum_{n,\ell=0}^{p-1}(-1)^{n+\ell}h(k)^{\langle\ell\rangle}h(m)^{\langle n\rangle}\otimes(1-e(k)t)^{-\ell}(1-e(m)t)^{-n}\\
&\quad\cdot d_m^{(n)}d_k^{(\ell)}((x^{(\alpha)}D_i)^p)t^{n+\ell}\\
&\qquad\qquad (\mathrm{mod}\ \ p,\ I_t\otimes U(\mathbf{W}(n;\underline{1}))[[t]]+U(\mathbf{W}(n;\underline{1}))[[t]]\otimes I_t)\\
&= (x^{(\alpha)}D_i)^p\otimes 1 + \sum_{n,\ell=0}^{p-1}(-1)^{n+\ell}h(k)^{\langle\ell\rangle}h(m)^{\langle n\rangle}\otimes(1-e(k)t)^{-\ell}(1-e(m)t)^{-n}\\
&\quad\cdot\big(\delta_{\ell,0}\delta_{n,0}(x^{(\alpha)}D_i)^p-(\delta_{n,0}\delta_{1,\ell}\delta_{ik}+\delta_{\ell,0}\delta_{1,n}\delta_{im})\,\delta_{\alpha,\epsilon_i}\,e(i)\big)t^{n+\ell}\\
&\qquad\qquad (\mathrm{mod}\ \ I_t\otimes U(\mathbf{W}(n;\underline{1}))[[t]]+U(\mathbf{W}(n;\underline{1}))[[t]]\otimes I_t)\\
&= (x^{(\alpha)}D_i)^p\otimes 1 + 1\otimes(x^{(\alpha)}D_i)^p + h(k)\otimes(1-e(k)t)^{-1}\delta_{ik}\delta_{\alpha,\epsilon_i}e(i)t\\
&\quad + h(m)\otimes(1-e(m)t)^{-1}\delta_{im}\delta_{\alpha,\epsilon_i}e(i)t.\\
&\qquad\qquad (\mathrm{mod}\ \ I_t\otimes U(\mathbf{W}(n;\underline{1}))[[t]]+U(\mathbf{W}(n;\underline{1}))[[t]]\otimes I_t).
\end{aligned}$$

Hence, when $\alpha\neq\epsilon_i$, we get

$$\begin{aligned}
\Delta((x^{(\alpha)}D_i)^p) &\equiv (x^{(\alpha)}D_i)^p\otimes 1+1\otimes(x^{(\alpha)}D_i)^p\\
&\subseteq I_t\otimes U(\mathbf{W}(n;\underline{1}))[[t]]+U(\mathbf{W}(n;\underline{1}))[[t]]\otimes I_t.
\end{aligned}$$

When $\alpha=\epsilon_i$, by Lemmas 3.4 and 3.10, (47) becomes $\Delta(x^{(\epsilon_i)}D_i)=x^{(\epsilon_i)}D_i\otimes 1+1\otimes x^{(\epsilon_i)}D_i+\delta_{ik}h(k)\otimes(1-e(k)t)^{-1}e(i)t+\delta_{im}h(m)\otimes(1-e(m)t)^{-1}e(i)t$. Combining with (53), we obtain

$$\begin{aligned}
\Delta((x^{(\epsilon_i)}D_i)^p-x^{(\epsilon_i)}D_i) &\equiv ((x^{(\epsilon_i)}D_i)^p-x^{(\epsilon_i)}D_i)\otimes 1+1\otimes((x^{(\epsilon_i)}D_i)^p-x^{(\epsilon_i)}D_i)\\
&\subseteq I_t\otimes U(\mathbf{W}(n;\underline{1}))[[t]]+U(\mathbf{W}(n;\underline{1}))[[t]]\otimes I_t.
\end{aligned}$$

Thereby, we prove that the ideal I_t is also a coideal of the Hopf algebra $U(\mathbf{W}(n;\underline{1}))[[t]]$.

(II) By Lemmas 3.9, 3.2, 3.4 & 3.10, we have

$$\begin{aligned}
S((x^{(\alpha)}D_i)^p) &= -(1-e(k)t)^{-p(\alpha_k-\delta_{ik})}(1-e(m)t)^{-p(\alpha_m-\delta_{im})}\\
&\qquad\cdot\Big(\sum_{n,\ell=0}^{\infty}d_m^{(n)}d_k^{(\ell)}((x^{(\alpha)}D_i)^p)\cdot h(k)_1^{\langle\ell\rangle}h(m)_1^{\langle n\rangle}t^{n+\ell}\Big)\\
&\equiv -\sum_{n,\ell=0}^{p-1}d_m^{(n)}d_k^{(\ell)}((x^{(\alpha)}D_i)^p)\cdot h(k)_1^{\langle\ell\rangle}h(m)_1^{\langle n\rangle}t^{n+\ell}\quad(\mathrm{mod}\ (p,I))\\
&= -(x^{(\alpha)}D_i)^p+\delta_{ik}\delta_{\alpha,\epsilon_i}e(i)\cdot h(k)_1^{\langle 1\rangle}t+\delta_{im}\delta_{\alpha,\epsilon_i}e(i)\cdot h(m)_1^{\langle 1\rangle}t.
\end{aligned}\tag{54}$$

Hence, when $\alpha\neq\epsilon_i$, we get

$$S\big((x^{(\alpha)}D_i)^p\big)\equiv -(x^{(\alpha)}D_i)^p\equiv 0\quad(\mathrm{mod}\,I_t).$$

When $\alpha=\epsilon_i$, by Lemmas 3.4 and 3.9, (48) reads as $S(x^{(\epsilon_i)}D_i)=-x^{(\epsilon_i)}D_i+\delta_{ik}e(i)\cdot h(k)_1^{\langle 1\rangle}t+\delta_{im}e(i)\cdot h(m)_1^{\langle 1\rangle}t$. Combining with (54), we obtain

$$S\big((x^{(\epsilon_i)}D_i)^p-x^{(\epsilon_i)}D_i\big)\equiv-\big((x^{(\epsilon_i)}D_i)^p-x^{(\epsilon_i)}D_i\big)\equiv 0\quad(\mathrm{mod}\,I_t).$$

Thereby, we verify that I_t is preserved by the antipode S of the quantization $U(\mathbf{W}(n;\underline{1}))[[t]]$ given in Lemma 3.8.

The proof is complete. □

Remark 3.12. Theorem 3.7 gives 2^n-1 new Hopf algebra structures of $\mathcal{K}$-dimension p^{np^n} over the same restricted universal enveloping algebra $\mathbf{u}(\mathbf{W}(n;\underline{1}))$ under the assumption that $\mathcal{K}$ is algebraically closed and that t is specialized at a root of the p-polynomial $t^p - q\,t$ with $q \in \mathcal{K}^*$.

ACKNOWLEDGMENT

NH would like to express his thanks to the ICTP for its support when he visited the ICTP Mathematics Group from March to August, 2006. Authors are indebted to David Hernandez, Leonard L. Scott for their kind help in improving our English writing, to the referee for his/her useful comments.

References

[1] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1995.
[2] D. Dokovic and K. Zhao, *Derivations, isomorphisms and second cohomology of generalized-Witt algebras*, Trans. Amer. Math. Soc. **350** (1998), 643–664.
[3] V.G. Drinfel'd, *Quantum groups*, Proceedings ICM (berkeley 1986) **1** (1987), AMS, 798–820.
[4] P. Etingof and O. Schiffmann, *Lectures on Quantum Groups*, 2nd, International Press, USA, 2002.
[5] A. Giaquinto and J. Zhang, *Bialgebra action, twists and universal deformation formulas*, J. Pure Appl. Algebra, **128** (2) (1998), 133–151.
[6] C. Grunspan, *Quantizations of the Witt algebra and of simple Lie algebras in characteristic p*, J. Algebra, **280** (2004), 145–161.
[7] W. Michaelis, *A class of infinite-dimensional Lie bialgebras containing the Virasoro algebras*, Adv. Math., **107** (1994), 365–392.
[8] S.H. Ng and E.J. Taft, *Classification of the Lie bialgebra structures on the Witt and Virasoro algebras*, J. Pure and Appl. Algebra, **151** (2000), 67–88.
[9] D.E. Radford, *Operators on Hopf algebras*, Amer. J. Math. **99** (1977), 139–158.
[10] G. Song and Y. Su, *Lie bialgeras of generalized-Witt type*, arXiv.Math: QA/0504168, Science in China, Ser. A—Math. **49** (4) (2006), 533–544.
[11] H. Strade, *Simple Lie Algebras over Fields of Positive Characteristic, I. Structure Theory*, de Gruyter Expositions in Mathematics, **38**, Walter de Gruyter, 2004.
[12] H. Strade and R. Farnsteiner, *Modular Lie Algebras and Their Representations*, Monogr. Textbooks, Pure Appl. Math. **116**, Marcel Dekker, 1988.
[13] E. Taft, *Witt and Virasoro algebras as bialgebras*, J. Pure Appl. Algebra **87** (3) (1993), 301–312.

Department of Mathematics, East China Normal University, Shanghai 200062, PR China

E-mail address: nhhu@math.ecnu.edu.cn

Department of Mathematics, East China Normal University, Shanghai 200062, PR China, and Department of Mathematics, Harbin Normal University, Harbin 150080, PR China

E-mail address: xlingwang@163.com